Analysis on graphs

Course given at TU Graz March 2013 **Prof. Dr. D. Lenz**

Exercise Sheet II **Due to a later point of time**

(1) Let (b, c) be a graph over the finite set X and L the associated operator. Then, $e^{-tL}1 \leq 1$ by general principles. Show the following:

(a) $e^{-tL}1 \neq 1$ for all (one) $t > 0$ if and only if $c \neq 0$.

(b) If (b, c) is connected, then $e^{-tL}1(x) < 1$ for all $x \in X$ whenever $c \neq 0$.

(2) Let (b, c) be a graph over the finite X with associated form Q and W_{eff} the effective resistance defined in the course. Show the following:

(a) $W_{eff}(x, y) = \max\{|f(x) - f(y)|^2 : Q(f) \leq 1\}$.

(b) The function ϱ with

$$\varrho(x, y) = W_{eff}^{1/2} \text{ for } x \neq y \text{ and } \varrho(x, y) = 0 \text{ for } x = y$$

is a metric on X.

(3) Let (b, c) be a graph over the finite X with associated operator L. Show the following formula relating the associated semigroup e^{-tL}, $t \geq 0$, and the associated resolvent $(L + \alpha)^{-1}$, $\alpha > 0$:

(a) $e^{-tL} = \lim_{n\to\infty} \left(\frac{n}{t}(L + \frac{n}{t})^{-1}\right)^n$ for any $t > 0$.

(b) $(L + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} dt$ for any $\alpha > 0$.

Hint: Show the formulae first for the case that L is just a number. Then, use your favorite form of spectral theorem to deal with e^{-tL} and $(L + \alpha)^{-1}$. For example you might use the decomposition $L = \sum_{j=1}^k \lambda_j E_j$ (with the different eigenvalues λ_j and the corresponding spectral projections E_j of L) to conclude (how?) that

$$(L + \alpha)^{-1} = \sum (\lambda_j + \alpha)^{-1} E_j$$

and

$$e^{-tL} = \sum e^{-t\lambda_j} E_j.$$