Spectral theory of discrete Operators - Notes[1]

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Contents

Introduction

In this chapter we discuss where we are coming from and where we are heading to. We consciously avoid rigorous terms at first to put through the big picture. All notions will be introduced rigorously in the chapters that follow.

1. Applied operator theory - The big picture

While the clarity of linear algebra stems from its restriction to studying linear maps on finite dimensional vector spaces, the beauty of analysis is expressed as saying that it is the 'art of taking limits'. Operator theory combines this clarity and beauty as it is concerned with the study of linear maps on infinite dimensional space. The fundamental observations for the development is that differentiation (and integration) are linear. The idea is to extend ideas from linear algebra to study functional equations such as the

(a) Schrödinger equation (stationary and time dependent),

(b) Heat equation.

They can be employed to treat real world problems coming from physics. We discuss these equations shortly. Let $\Omega \subseteq \mathbb{R}^d$ be open,

$$\Delta = \sum_{i=1}^{d} \partial_i^2$$

the Laplacian and let $V \colon \Omega \to \mathbb{R}$ be a potential. (More generally we let Ω be a Riemannian manifold and Δ the Laplace-Beltrami operator.)

1.1. The stationary Schrödinger equation. In quantum mechanics the state of a quantum mechanical particle (e.g. an electron) is described by a function $f \colon \Omega \to \mathbb{C}$ with $\int_\Omega |f|^2 dx = 1$, a so-called wave function. The probabilty that the particle in state f can be found in $A \subseteq \Omega$ is interpreted as

$$\int_A |f|^2 dx.$$

Solutions to the stationary Schrödinger equation describe one quantum mechanical particle in a given medium that does not evolve in time. More precisely, for $E \in \mathbb{R}$ a wave function that solves

$$-\Delta f + Vf = Ef \tag{$\heartsuit$}$$

describes a particle in state f as follows:

(a) The operator Δ encodes the kinetic energy of the particle.

(b) The potential V encodes the potential energy of the particle. It depends on the medium.

(c) $E \in \mathbb{R}$ is the total energy of the particle in state f. An integration by parts formula (Green's formula) shows

$$E = \int_\Omega Ef^2 dx = \int_\Omega (-\Delta f + Vf) f dx = \int_\Omega |\nabla f|^2 dx + \int_\Omega |f|^2 V dx.$$

The mapping $H\colon f \mapsto -\Delta f + Vf$ is called Schrödinger operator. It is obviously linear and so Equation ($\heartsuit$), which reads $Hf = Ef$, is an eigenvalue equation. As seen above, the functional

$$Q\colon f \mapsto \int_\Omega |\nabla f|^2 dx + \int_\Omega |f|^2 V dx$$

also describes the energy of a particle. It is a quadratic form. We shall see below that operators and energy functionals (quadratic forms) are indeed intimatley linked.

The set of all $E \in \mathbb{R}$ for which Equation ($\heartsuit$) has a generalized solution is the set of possible energies that the particle can assume. It is called the spectrum of H and denoted by $\sigma(H)$. It is alternatively described by the closure of the set $\{Q(f) \mid \int_\Omega |f|^2 dx = 1\}$.

It is of interest to determine the spectrum $\sigma(H)$. The quantity $\lambda_0(H) = \inf \sigma(H)$, the smallest possible E for which Equation ($\heartsuit$) has a generalized solution, is the so-called ground-state energy. Properties of the corresponding generalized solution are important, because they determine the long-time behaviour of solutions to the heat equation (see below).

1.2. Time-dependent Schrödinger equation. For a particle in a given state f_0 its time evolution is modeled as follows. Let $f\colon \mathbb{R}\times\Omega \to \mathbb{C}$ (smooth enough) with $\int_\Omega |f(t,x)|^2 dx = 1$ for all $t \in \mathbb{R}$ such that

$$i\partial_t f = Hf, \quad f(0,\cdot) = f_0.$$

Then $f(t,\cdot)$ is the state of the particle after time t provided it was in state f_0 at time $t = 0$. Formally one can solve this Schrödinger equation by letting

$$f(t,\cdot) = e^{itH} f_0.$$

If f_0 is a normalized solution ($\int_\Omega |f_0|^2 dx = 1$) to the stationary Schrödinger equation $Hf_0 = Ef_0$, then $g(t,\cdot) = e^{itE} f_0$ is a solution to the the time-dependent Schrödinger equation. Since for $A \subseteq \Omega$ we have

$$\int_A |g(t,x)|^2 dx = \int_A |e^{itE}|^2 |f_0|^2 dx = \int_A |f_0|^2 dx,$$

the particle in such a state does not evolve in time, it is localized. This is why Equation (♡) is called stationary Schrödinger equation and such a state is called bound state.

Now assume that $E \in \mathbb{R}$ is such that Equation (♡) does not have a solution. Let f_0 be normalized ($\int_\Omega |f_0|^2 dx = 1$) with energy $E = Q(f_0) \in \mathbb{R}$ and let f be a corresponding solution to the time-dependent Schrödinger equation. Then (it is very likely that) for every compact $K \subseteq \Omega$ we have

$$\int_K |f(t,x)|^2 dx \to 0 \text{ as } t \to \infty.$$

Hence, the particle leaves every compact set and so it is delocalized.

Since solutions to both Schrödinger equations govern the behavior of quantum mechanical Systems, it is very important to investigate whether or not they are unique. If they are not unique, then further physical assumptions have to be made in order to determine 'correct solutions'.

1.3. The heat equation. Heat is measured by a nonnegative function $f\colon \Omega \to \mathbb{R}$, where the total amount of heat in a subset $A \subseteq \Omega$ is described by $\int_A f dx$. A solution $g\colon [0,\infty) \times \Omega \to \mathbb{R}$ to the equation

$$\partial_t f = Hf, \quad g(0,\cdot) = f$$

describes the heat at time t through $g(t,\cdot)$ provided the heat at time $t = 0$ was f.

The Operator $-\Delta$ models diffusion, at places $x \in \Omega$ with $V(x) > 0$ heat is drained from the system while at places with $V(x) < \infty$ heat is inserted to the system. Formally the heat equation can be solved by letting

$$g(t,\cdot) = e^{-tH} f.$$

In view of the interpretation as heat, it is an important property that $f \geq 0$ implies $g(t,\cdot) \geq 0$.

Spectral theory of H determines the long-time behavior of solutions to the heat equation. More precisely,

$$e^{-tH} f = e^{-t\lambda_0(H)} \left(Pf + e^{-t(\lambda_1(H) - \lambda_0(H))} Q_t f \right).$$

Here P is the projection to the space spanned by the ground state, $\lambda_1(H) = \inf \sigma(H) \setminus \{\lambda_0(H)\}$ and Q_t is a uniformly bounded operator (with uniformly bounded norm in t). The quantity $\lambda_1(H) - \lambda_0(H)$ is called spectral gap. It determines how quick the heat flow converges to the ground state.

There are several other relevant quantities. Suppose that $f \geq 0$ solves the heat equation. For some compact $K \subseteq \Omega$ the integral

$$\int_0^\infty \int_K f dx dt$$

measures the total amount of heat moved through K over time. It is an interesting question whether or not this quantity is finite or not as this has many (spectral) consequences. It is discussed under the name recurrence v.s. transience.

The total amount of heat at time t is

$$\int_\Omega f(t,x)dx.$$

It is an important question whether or not this quantity is constant over time. This is discussed under the name conservativeness or stochastic completeness.

All the questions discussed in this section depend on the potential V and the geometry of Ω.

2. Discrete Operators

Studying the previously discussed equations is in some sense hard on a technical level. Treating partial differential equations requires a lot of knowledge on distributions, function spaces, local regularity theory etc. Since they are simplified models for real world problems, there is no reason why one should not simplify further. Discrete models have the advantage that on a discrete space all functions are continous and so (most) of the regularity issues disappear. It turns out that the qualitative behavior of solutions to the discrete Schrödinger equation and the discrete heat equation is the same as in the continuum. However, besides less accurate physical results, there is another price that one has to pay. Differential operators have good algebraic properties (product rule, chain rule etc.), while discrete operators often lack this structure.

Next we discuss how to properly discretize H. The action of V by multiplication is the same as in the continuum. Hence, we can restrict the attention to Δ. Suppose now that we cannot access function values at all points of Ω but only at a grid $G := \Omega \cap h\mathbb{Z}^d$ of scale $h > 0$. For small $h > 0$ and $x \in G$ we then have

$$\partial_i f(x) \approx \frac{f(x + he_i) - f(x)}{h}$$

and

$$\partial_i f(x) \approx \frac{f(x) - f(x - he_i)}{h},$$

where e_i is the i-th unit vector. Plugging this into the formula for the second partial derivative we arrive at

$$\begin{aligned}\partial_i^2 f(x) &\approx \frac{\partial_i f(x + he_i) - \partial_i f(x)}{h} \\ &\approx \frac{\frac{f(x+he_i) - f(x+he_i-he_i)}{h} - \frac{f(x)-f(x-he_i)}{h}}{h} \\ &= \frac{f(x + he_i) + f(x - he_i) - 2f(x)}{h^2}.\end{aligned}$$

Indeed, it follows directly from Taylor's theorem that the right-hand side of the previous equation converges to $\partial_i^2 f(x)$. Summing up shows

$$\Delta f(x) \approx \sum_{i=1}^{d} \sum_{\sigma \in \{-1,1\}} \frac{f(x + \sigma he_i) - f(x)}{h^2}.$$

On G we define the relation $\sim$ as $x \sim y$ if and only if $|x - y| = h$, then this formula reads

Zeichnung

$$\Delta f(x) \approx \sum_{y \in G,\, y \sim x} \frac{1}{h^2}(f(y) - f(x)).$$

We interpret $(G, \sim)$ as a graph with edges of weight h^{-2} between points x, y with $x \sim y$. Below we will study this type of operator for more general graphs and more general weights. More precisely, we will study the following questions raised above for the discrete operators associated with infinite graphs:

(a) Existence and uniqueness of self-adjoint realizations.

(b) Lower bounds for the bottom of the (essential) spectrum.

(c) Recurrence v.s. transience

(d) Stochastic completeness

CHAPTER 1

Preliminaries - Quadratic forms, associated operators and spectral theory

In this chapter we intrduce closed quadratic forms and discuss how they give rise to self-adjoint operators. In what follows we always let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and denote by $\|\cdot\|$ the corresponding norm. The Hilbert space that we will mainly consider below is the following.

Example. Let X be a countable set and let $m\colon X \to (0,\infty)$. Then

$$\ell^2(X,m) := \{f\colon X \to \mathbb{R} \mid \sum_{x\in X} |f(x)|^2 m(x)\}$$

equipped with the scalar product

$$\langle f, g\rangle := \sum_{x\in X} f(x)g(x)m(x)$$

is a real Hilbert space. Of further interest are for $1 \leq p < \infty$ the Banach spaces

$$\ell^p(X,m) := \{f\colon X \to \mathbb{R} \mid \sum_{x\in X} |f(x)|^p m(x)\}$$

with norm

$$\|f\|_p := \left(\sum_{x\in X} |f(x)|^p m(x)\right)^{1/p}$$

and the Banach space

$$\ell^\infty(X) := \{f\colon X \to \mathbb{R} \mid \sup_{x\in X} |f(x)| < \infty\}$$

with norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

Remark. The restriction to real Hilbert spaces does not affect the generality of our results, but it makes certain arguments and formulas shorter. The discrete Schrödinger operators that we study below are so-called real-operators on $\ell^2(X,m)$. Their properties when viewed as operators on the complex Hilbert space $\ell^2_{\mathbb{C}}(X,m) = \{f + ig \mid f, g \in \ell^2(X,m)\}$ are uniquely determined by their properties as operators on $\ell^2(X,m)$.

1. Closed forms

Definition (Quadratic form)**.** A *quadratic form* on H is a functional

$$q\colon H \to (-\infty, \infty]$$

with the following properties:

(a) $q(\lambda f) = |\lambda|^2 q(f)$ for all $f \in H$, $\lambda \in \mathbb{R}$. (Homogeneity)

(b) $q(f+g) + q(f-g) = 2q(f) + 2q(g)$ for all $f, g \in H$.
(Parallelorgramm Identity)

In this case, the set $D(q) := \{f \in H \mid q(f) < \infty\}$ is called the *domain* of q and $\ker(q) := \{f \in H \mid q(f) = 0\}$ is called the *kernel* of q. The form q is called *densely defined* if $D(q)$ is dense in H.

Remark. Here we use the convention $0 \cdot \infty = 0$ and $a + \infty = \infty$ whenever $a \in (-\infty, \infty]$.

Exercise 1. If q is a quadratic form, then $D(q)$ is a vector space.

Every quadratic form q on H induces a symmetric bilinear form $\tilde{q}$ on $D(q)$ via polarization. More precisely, the functional

$$\tilde{q}\colon D(q) \times D(q) \to \mathbb{R}, \quad \tilde{q}(f,g) = \frac{1}{4}\left(q(f+g) - q(f-g)\right)$$

is symmetric and bilinear, i.e. $\tilde{q}(f,g) = \tilde{q}(g,f)$ and $\tilde{q}(\lambda f + \mu g, h) = \tilde{\lambda} q(f,h) + \mu q(g,h)$ for $\lambda, \mu \in \mathbb{R}$ and $f, g, h \in H$, see [**1**].

Exercise 2. A bilinear form s definied on a linear subspace $D(s) \subseteq H$ induces a quadratic form

$$q_s\colon H \to (-\infty, \infty], \quad q_s(f) = \begin{cases} s(f,f) & \text{if } f \in D(s) \\ \infty & \text{else} \end{cases}.$$

It satisfies $D(q_s) = D(s)$ and $\widetilde{q_s}(f,g) = s(f,g)$ for $f, g \in D(s)$.

The previous exercise shows that quadratic forms on H and bilinar forms with domain are in a $1:1$ relationship. Hence, for a quadratic form q on H we abuse notation and just write q for the induced bilinear form $\tilde{q}$. In this sense, we have $q(f) = q(f,f)$ whenever $f \in D(q)$.

Exercise 3. (a) The functional $\|\cdot\|^2\colon H \to [0, \infty)$ is a quadratic form on H with $D(\|\cdot\|^2) = H$ and $\ker(\|\cdot\|^2) = \{0\}$. The induced bilinear form is given by $\langle \cdot, \cdot \rangle$.

(b) If $V\colon X \to \mathbb{R}$ with $V_- = \max\{-V, 0\} \in \ell^\infty(X)$, then

$$q_V\colon \ell^2(X,m) \to (-\infty, \infty], \quad q_V(f) = \sum_{x \in V} |f(x)|^2 V(x) m(x)$$

is a quadratic form. Its domain is given by $D(q_V) = \ell^2(X,m) \cap \ell^2(X,|V|m)$ and the induced bilinear form is given by

$$q_V(f,g) = \sum_{x \in V} f(x)g(x)V(x)m(x).$$

Definition (Lower semibounded form)**.** A quadratic form q on H is called *lower semibounded* with lower bound $b \in \mathbb{R}$ if

$$q(f) \geq b\|f\|^2$$

for all $f \in H$. The largest possible lower bound for q is denoted by $\lambda_0(q)$. If 0 is a lower bound for q, i.e. $q(f) \geq 0$ for all $f \in H$, then q is called *nonnegative*.

It follows from the definition of $\lambda_0(q)$ that

$$\begin{aligned}\lambda_0(q) &= \sup\{b \in \mathbb{R} \mid q(f) \geq b\|f\|^2 \text{ for all } f \in H\} \\ &= \sup\{b \in \mathbb{R} \mid q(f) \geq b\|f\|^2 \text{ for all } f \in D(q)\} \\ &= \inf\left\{\frac{q(f)}{\|f\|^2} \,\middle|\, f \in H \setminus \{0\}\right\}.\end{aligned}$$

To $\alpha \in \mathbb{R}$ we associate the quadratic form

$$q_\alpha \colon H \to (-\infty, \infty], \quad q_\alpha(f) = q(f) + \alpha\|f\|^2.$$

Its domain satisfies $D(q_\alpha) = D(q)$ and the associated bilinear form is given by

$$q_\alpha(f,g) = q(f,g) + \alpha\langle f, g\rangle.$$

Lemma (Properties of q_α)**.** *Let q be a lower semibounded quadratic form on H. If $\alpha \geq -\lambda_0(q)$, then q_α is nonnegative. If $\alpha > -\lambda_0(q)$, then* $\ker(q_\alpha) = \{0\}$ *so that q_α is a scalar product on $D(q)$. In this case, the embedding*

$$(D(q), q_\alpha) \to (H, \langle\cdot,\cdot\rangle), \quad f \mapsto f$$

is continuous.

Proof. All these statements follow from the inequality

$$q_\alpha(f) \geq (\alpha + \lambda_0(q))\|f\|^2, \quad f \in D(q).$$

□

Exercise 4. Show that for any $\alpha, \beta > -\lambda_0(q)$ the norms on $D(q)$ induced by the scalar products q_α and q_β are equivalent.

Exercise 5. Let $\alpha > -\lambda_0(q)$. Show that if $D \subseteq D(q)$ is dense with respect to q_α, then

$$\lambda_0(q) = \inf\left\{\frac{q(f)}{\|f\|^2} \,\middle|\, f \in D \setminus \{0\}\right\}.$$

Exercise 6. Show that if q is nonnegative, then $\ker(q)$ is a vector space. Give an example that this need not be true for general quadratic forms.

Hint: Cauchy-Schwarz.

We recall the following compactness result in Hilbert spaces.

Theorem (Banach-Alaouglu-Bourbaki and Banach-Saks)**.** *Let (f_n) be a bounded sequence in H.*

(a) It has a subsequence that convergences weakly in H.

(b) It has a subsequence (f_{n_k}) whose sequence of Césaro means

$$\left(\frac{1}{N}\sum_{k=1}^{N} f_{n_k}\right)_{N\geq 1}$$

converges in H.

Proof. (a): Without loss of generality we can assume that H is seprarable (else split $H = \overline{\mathrm{Lin}\,\{f_n \mid n \in \mathbb{N}\}} \oplus \mathrm{Lin}\,\{f_n \mid n \in \mathbb{N}\}^{\perp}$ and use that $\langle f_n, g\rangle = 0$ for all $g \in \mathrm{Lin}\,\{f_n \mid n \in \mathbb{N}\}^{\perp}$). Let $\{g_n \mid n \in \mathbb{N}$ be a dense subset of H. By a diagonal sequence argument we choose a subsequence (f_{n_k}) of (f_n) such that for each $m \in \mathbb{N}$ the sequence $\langle g_m, f_{n_k}\rangle$ converges, as $k \to \infty$. It follows that $\langle g, f_{n_k}\rangle$ converges for all $g \in H$. We define the linear functional

$$\varphi\colon H \to \mathbb{R},\ \varphi(g) = \lim_{k\to\infty} \langle g, f_{n_k}\rangle.$$

The boundedness of (f_n) implies that φ is bounded. By the Riesz representation theorem there exists an $f \in H$ such that $\varphi(g) = \langle g, f\rangle$. This proves $f_{n_k} \to f$ weakly in H.

(b): Using (a) without loss of generality we can assume that (f_n) converges weakly to 0. We inductively choose a subsequence (f_{n_k}) such that for each $N \geq 0$ we have

$$|\langle f_{n_1}, f_{n_{N+1}}\rangle| \leq \frac{1}{N}, \dots, |\langle f_{n_N}, f_{n_{N+1}}\rangle| \leq \frac{1}{N}.$$

We denote by M a bound for $\|f_n\|$. The Césaro mean g_N of (f_{n_k}) satisfies

$$\begin{aligned}\|g_N\|^2 &= \frac{1}{N^2}\sum_{k=1}^{N}\|f_{n_k}\|^2 + \frac{2}{N^2}\sum_{1\leq i<k\leq N}\langle f_{n_i}, f_{n_k}\rangle \\ &\leq \frac{M^2}{N} + \frac{2}{N^2}\sum_{k=1}^{N}\frac{k-1}{k} \leq \frac{M^2+2}{N}.\end{aligned}$$

Letting $N \to \infty$ shows $g_N \to 0$ and the claim is proven. □

Exercise 7. Prove that a sequence (f_n) in H that is weakly convergent to $f \in H$ has a subsequence whose sequence of Césaro means converges in H to f.

Hint: Uniform boundedness principle.

The following proposition is very important for further considerations.

Proposition (Characterization of closed forms)**.** *Let q be a lower semi-bounded quadratic form on H. The following assertions are equivalent.*

(i) For one/any $\alpha > -\lambda_0(q)$ the space $(D(q), q_\alpha)$ is a Hilbert space.

(ii) q is lower semicontinuous, i.e. for any sequence (f_n) and f in H with $f_n \to f$ in H we have

$$q(f) \leq \liminf_{n\to\infty} q(f_n).$$

Proof. (ii) $\Rightarrow$ (i): Let $\alpha > -\lambda_0(q)$ and let (f_n) be a Cauchy sequence in $(D(q), q_\alpha)$. With $\beta := \alpha + \lambda_0(q)$ we have

$$q_\alpha(f_n - f_m) \geq (\alpha + \lambda_0(q))\|f_n - f_m\|^2 = \beta\|f_n - f_m\|^2.$$

Since $\beta > 0$, this implies that (f_n) is Cauchy with respect to $\|\cdot\|$. Hence, it has a limit $f \in H$. The lower semicontinuity of q shows

$$q(f - f_n) \leq \liminf_{m\to\infty} q(f_n - f_m) < \infty,$$

where the finiteness of the right hand side follows from the fact that (f_n) is q_α-Cauchy and $\|\cdot\|$-Cauchy. Since $f_n \in D(q)$ and $D(q)$ is a vector space, we obtain $f \in D(q)$. Moreover, the above inequality yields

$$q_\alpha(f - f_n) \leq \liminf_{m\to\infty} q_\alpha(f_n - f_m) \to 0, n \to \infty,$$

since (f_n) is q_α-Cauchy.

(i) $\Rightarrow$ (ii): After passing to a suitable subsequence we can assume

$$\lim_{n\to\infty} q(f_n) = \liminf_{n\to\infty} q(f_n) < \infty,$$

for otherwise there is nothing to show. It follows from this q-boundedness of (f_n) and the boundedness of (f_n) in H that (f_n) is bounded in the Hilbert space $(D(q), q_\alpha)$. By the Banach-Saks theorem it has a subsequence (f_{n_k}) whose sequence of Césaro means

$$g_N = \frac{1}{N}\sum_{k=1}^{N} f_{n_k}$$

converges in $(D(q), q_\alpha)$ to some $g \in D(q)$. Since convergence with respect to q_α implies convergence in H, we obtain $g_N \to g$ in H. Moreover, $f_n \to f$ in H implies $g_N \to f$ in H. Since limits are unique, we arrive at $f = g \in D(q)$. Using this observation and that the square root of q_α is a norm, we infer

$$q_\alpha(f)^{1/2} = \lim_{N\to\infty} q_\alpha(g_N)^{1/2} \leq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} q_\alpha(f_{n_k})^{1/2} = \lim_{n\to\infty} q_\alpha(f_n)^{1/2}.$$

From this inequality and the convergence $\|f_n\|^2 \to \|f\|^2$ the claim follows. □

Definition. A lower semibounded quadratic form q on H is called *closed*, if it satisfies one of the conditions of the previous lemma.

Example. Let $V\colon X \to \mathbb{R}$ wit $V_- \in \ell^\infty(X)$ and let q_V be the quadratic form on $\ell^2(X, m)$ discussed above. Then q_V is lower semibounded with $\lambda_0(q_V) = \inf_{x\in X} V(x) = -\sup_{x\in X} V_-(x)$ and closed.

Proof. ... □

Definition (Extension and restriction)**.** We say that a quadratic form q' is an *extension* of q if $D(q') \supseteq D(q)$ and $q(f) = q'(f)$ for $f \in D(q)$. For a subspace $V \subseteq D(q)$ we define the *restriction* $q|_V$ of q to V by

$$q|_V\colon H \to (-\infty, \infty], \quad q_V(f) = \begin{cases} q(f) & \text{if } f \in V \\ \infty & \text{else} \end{cases}.$$

Exercise 8. Show that if q' is an extension of q, then $\lambda_0(q') \leq \lambda_0(q)$.

Often forms are not closed but have one (or many) closed extensions. This can be characterized as follows.

Proposition (Characterization of closable forms)**.** *Let q be a lower semibounded quadratic form on H. The following assertions are equivalent.*

(i) q has a closed extension.

(ii) For every sequence (f_n) and f in $D(q)$ the convergence $f_n \to f$ in H implies

$$q(f) \leq \liminf_{n\to\infty} q(f_n).$$

In this case, q has a smallest closed extension $\bar{q}$, i.e. every other extension of q is also an extension of $\bar{q}$. Let $\alpha > -\lambda_0(q)$. Then $\bar{q}$ is given by

$$\bar{q}(f) = \begin{cases} \lim\limits_{n\to\infty} q(f_n) & \text{if } (f_n) \text{ is } q_\alpha\text{-Cauchy with } f_n \to f \text{ in } H \\ \infty & \text{else} \end{cases}.$$

It satisfies $\lambda_0(\bar{q}) = \lambda_0(q)$.

Remark. The characterization of closedness (ii) and closability (ii) look very similar. The only difference is that in the characterization of closability the function f is assumed to be in $D(q)$, while in the characterization of closedness $f \in D(q)$ has to be implied by the q-boundedness of (f_n).

Proof. (i) $\Rightarrow$ (ii): This follows from the characterization of closed forms.

(ii) $\Rightarrow$ (i): Exercise.

If q' is a closed extension of q, then so is the restriction of q' to the closure of $D(q)$ with respect to q'_α for some $\alpha > -\lambda_0(q')$. It is readily verified that this closure is exactly given by the claimed formula. The formula for λ_0 follows from Exercise 5. □

Exercise* 9. Proof the implication (ii) $\Rightarrow$ (i) of the previous proposition.

Hint: It suffices to prove that the form q' defined by

$$q'(f) = \begin{cases} \lim\limits_{n\to\infty} q(f_n) & \text{if } (f_n) \text{ is } q_\alpha\text{-Cauchy with } f_n \to f \text{ in } H \\ \infty & \text{else} \end{cases}$$

is well-defined and closed.

Example. Consider the Hilbert space $L^2((-1,1))$ with respect to the Lebesgue measure. We define the quadratic form

$$q(f) = \begin{cases} |f(0)|^2 & \text{if } f \in C((-1,1)) \\ \infty & \text{else} \end{cases}.$$

It is not closable.

Proof. Consider a sequence of continuous functions (f_n) with $f_n \to 1$ in $L^2((-1,1))$ but with $f_n(0) = 0$. One example for such a sequence is given by

$$f_n(x) = \begin{cases} 1 & \text{if } n^{-1} \le |x| < 1 \\ n|x| & \text{if } |x| \le n^{-1} \end{cases}.$$

Then we have

$$q(1) = 1 > 0 = \lim_{n\to\infty} |f_n(0)|^2 = \lim_{n\to\infty} q(f_n),$$

which together with the previous characterization of closability shows that q is not closable. □

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[1] Dirk Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2007.