Analysis on graphs - Notes [1]

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[1] This is a rather uncorrected version. Helpful comments are welcome.

Contents

CHAPTER 1

Heat equation on finite sets

Introduction: Heat equation on $\Omega \subset \mathbb{R}^N$ and Markov property

The heat equation on the open subset $\Omega \subset \mathbb{R}^N$ reads as

$$\frac{d}{dt}\psi_t = \Delta\psi_t, \ \psi_0 = f.$$

Here, $\psi = \psi_t(x) = \psi(t,x)$ depends on the time variable $t \geq 0$ and the space variable $x \in \Omega$. It describes the evolution over time of the density of a diffusion process. Examples are diffusion of heat in space, of ink in a glass of water, ...

To set a perspective we are going to derive the heat equation next: The amount of 'heat' in the volume V at the time t is given as

$$\int_V \psi(t,x)dx.$$

The change of this amount can be computed in two different ways as follows. One way to compute it is to just take the time derivative. This leads to

$$\text{Change} = \frac{d}{dt}\int_V \psi(t,x)dx = \int_V \frac{d}{dt}\psi(t,x)dx.$$

On the other hand, the change is due to flow. This leads to

$$\begin{aligned} \text{Change} &= -\int_{\partial V} (flow)\cdot n dS \\ &= \int_{\partial V} B_x \nabla\psi_t \cdot n dS \\ (Stokes) \quad &= \int_V \nabla\cdot(B_x\nabla)\psi_t dx. \end{aligned}$$

(First equation: Change due to flow; n outer normal, sign due to measuring outflow. Second equation: Here enters the physical model for diffusion: flow due to gradient of $-\psi$ 'from warm to cold', dependence of gradient linear with possible dependence on position, this leads to $B_x \geq 0$. Third equation: Stokes / Gauss theorem.) Setting the two terms giving the change equal then leads to

$$\int_V \frac{d}{dt}\psi(t,x)dx = \int_V \nabla\cdot(B_x\nabla)\psi_t dx.$$

As this holds for arbitrary (smooth) V, we infer

$$\frac{d}{dt}\psi_t = \nabla\cdot(B_x\nabla)\psi_t.$$

This is the general form of the heat equation.

If the medium is homogenous, then B does not depend on x i.e. $B \equiv$ constant matrix. If then futhermore the medium is isotropic, then there is no preferred direction and the matrix is the identity. This gives then

$$\frac{d}{dt}\psi_t = \Delta\psi_t.$$

This is the heat equation in the form given above. At least formally a solution is given by

$$\psi_t = e^{t\Delta}f \quad (\psi' = \Delta\psi, \psi_0 = f).$$

The family $e^{t\Delta}, t \geq 0$, is called the *semigroup* associated to the Laplacian or just the *heat semigroup*. In line with basic ideas on diffusion it has the following *Markov property*

$$0 \leq e^{t\Delta}f \text{ for all (suitable) } f \geq 0$$

and

$$e^{t\Delta}f \leq 1 \text{ for all (suitable) } f \leq 1.$$

Note that both inequalities really make sense for a diffusion model from a physical point of view.
A main question then concerns the long term behaviour of ψ. This leads to investigation of spectral theory of Δ.

1. Markov property and heat equation on a finite set

In this section we will consider the heat equation on a finite set. More specifically, we will characterize those operators which can be seen as suitable replacements of Laplacians in that they give rise to semigroups satisfying the Markov property. This will be expressed via the two Beurling / Deny criteria.

Here is the framework we consider. Set $X = \{1, \ldots, N\}$. The associated Hilbert space is

$$\mathcal{H} = \ell^2_{\mathbb{R}}(X) = \{f : X \longrightarrow \mathbb{R}\} = \mathbb{R}^N$$

with inner product

$$\langle f, g\rangle := \sum_{x\in X} f(x)g(x)$$

and induced norm

$$\|f\| := \langle f, f\rangle^{1/2}.$$

Of course, the norm $\|A\|$ of an linear operator from $\mathcal{H}$ into itself is then given by

$$\|A\| := \max\{\|Af\| : \|f\| \leq 1\}.$$

This is indeed a norm and makes the space of all linear operators from $\mathcal{H}$ into itself into a complete space. Moreover, it it submultiplicative i.e. $\|AB\| \leq \|A\|\|B\|$ holds.

Let $L : \mathcal{H} \longrightarrow \mathcal{H}$ be a selfadjoint operator (i.e. L is a linear operator satisfying $\langle Lf, g\rangle = \langle f, Lg\rangle$ for all $f, g \in \mathcal{H}$). Then, L is represented by the matrix $(l(x, y))$ satisfying

$$Lf(x) = \sum_{y\in X} l(x, y)f(y)$$

for all $f \in \mathcal{H}$). Selfadjointness of L and the fact that L maps real functions to real functions then just means that

$$l(x,y) = l(y,x), l(x,y) \in \mathbb{R} \text{ and for all } x,y \in X.$$

To L we associate the *form* Q defined by

$$Q(f,g) = \langle Lf, g\rangle = \langle f, Lg\rangle.$$

We use the notation

$$Q(f) := Q(f,f).$$

The 'heat equation' associated to L now reads

$$\frac{d}{dt}\psi_t = -L\psi_t \;\; \psi_0 = f.$$

The solution to this equation is given by

$$\psi_t = e^{-tL} f.$$

Here, e^{-tL} is defined by the power series

$$e^{-tL} = \sum_{n\geq 0} \frac{1}{n!}(tL)^n.$$

(To check this is left as an exercise in power series for the exponential function. One bascially needs to show the following:

- For any square matrix A the series $\sum_{n\geq 0} \frac{1}{n!} A^n$ converges absolutely, as $\sum_{n\geq 0} \frac{1}{n}\|A\|^n = e^{\|A\|} < \infty$.
- The map $t \mapsto e^{-tL}$ is differentiable with derivative given by $-Le^{-tL} = e^{-tL}L$.)

We call $e^{-tL}, t \geq 0$, the *semigroup associated to* L. In order to be able to consider L as a suitable operator for a heat equation we want the semigroup to have the *Markov property*. More precisely, we want the following to hold:

- $f \geq 0$ implies $e^{-tL} f \geq 0$ for all $t \geq 0$ ('the semigroup is *positivity preserving*')
- $f \leq 1$ implies $e^{-tL} f \leq 1$ for all $t \geq 0$ ('the semigroup is *contracting*')

If the semigroup is both positivity preserving and contracting it is called *Markov semigroup*.

In order to study the Markov property it is useful to consider another representation of L involving differences. This is discussed next.
A direct computation shows that

$$Lf(x) = \sum_{y\in X} b(x,y)(f(x) - f(y)) + c(x)f(x)$$

with

$$b(x,y) = -l(x,y), \;\text{ for } x \neq y \;\text{ and }\; b(x,x) = 0$$

and

$$c(x) = \sum_{z\in X} l(x,z)$$

for all $x, y \in X$. In terms of b, c the form Q can be expressed as

$$Q(f,g) = \frac{1}{2} \sum_{x,y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c(x)f(x)g(x)$$

(as can be checked by direct computation, which is left as an exercise).

Remark. The form associated to the operator $\nabla \cdot B\nabla$ considered above is given by

$$\langle \nabla \cdot B\nabla f, g\rangle = \langle B\nabla f, \nabla g\rangle.$$

This is structurally very similar to our expression for Q and this is not a coincidence (see later).

From now on we will mostly think about L and Q in terms of b and c.

THEOREM. *(1. Beurling / Deny criterion) Let $b : X\times X \longrightarrow \mathbb{R}$ be symmetric with vanishing diagonal and $c : X \longrightarrow \mathbb{R}$ be arbitrary and let L and Q be the associated operator and form. Then, the following assertions are equivalent:*

(i) $b \geq 0$ *('operator L')*
(ii) $Q(|f|) \leq Q(f)$ *for all f* *('form')*
(iii) $e^{-tL} f \geq 0$ *for all $f \geq 0$* *('semigroup')*

Remarks.

- The function c does not play a role in this theorem.
- The condition $b \geq 0$ in (i) can also be expressed as $l(x,y) \leq 0$ for all x, y with $x \neq y$ (as $b(x,y) = -l(x,y)$ for such x, y).
- The property in (ii) can be characterized differently (Exercise):

$$Q(|f|) \leq Q(f) \Longleftrightarrow Q(f_+, f_-) \leq 0 \text{ for all } f.$$

  Here, $f_+ := \max\{f, 0\}$, $f_- := \max\{-f, 0\}$ (and then $f_+, f_- \geq 0$, $f = f_+ - f_-, |f| = f_+ + f_-$.)
- The property in (iii) can be characterized differently:

$$e^{-tL} f \geq 0 \text{ for } f \geq 0 \Longleftrightarrow |e^{-tL} f| \leq e^{-tL}|f| \text{ for all } f.$$

  ($\Longrightarrow$: $f \geq 0$ implies $f = |f|$ and this gives then $e^{-tL} f = e{-}tL|f| \geq |e^{-tL} f| \geq 0$.
  $\Longleftarrow$: $f = f_+ - f_-, |f| = f_+ + f_-$. Thus,

$$\begin{aligned} |e^{-tL} f| &= |e^{-tL} f_+ - e^{-tL} f_-| \\ &\leq |e^{-tL} f_+| + |e^{-tL} f_-| \\ &= e^{-tL} f_+ + e^{-tL} f_- \\ &= e^{-tL}|f|. \end{aligned}$$

  Here, we used the assumption in the previous to the last step.)

PROOF. (i)$\Longrightarrow$ (iii): Recall Lie-Trotter-product formula (exercise)

$$e^{A+B} = \lim_{n\to\infty} (e^{\frac{1}{n}A} e^{\frac{1}{n}B})^n.$$

(Sketch of Proof: Set $S_n := e^{\frac{1}{n}(A+B)}$, $T_n := e^{\frac{1}{n}A}e^{\frac{1}{n}B}$. Then a telescoping argument gives

$$S_n^n - T_n^n = \sum_{m=0}^{n-1} S_n^m (S_n - T_n) T_n^{n-1-m}$$

hence

$$\|S_n^n - T_n^n\| \leq c_1 n \|S_n - T_n\|.$$

Moreover

$$\|S_n - T_n\| = \left\| \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{C+D}{n}\right)^m - \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{C}{n}\right)^k \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{D}{n}\right)^l \right\| \leq c_2 \frac{1}{n^2}.$$

This yields the claim.)

We set

$$L = \widetilde{L} + \widetilde{D}$$

with $\widetilde{L}$ coming from L be setting the diagonal to zero and $\widetilde{D}$ coming from L by setting the off-diagonals zero. Then, Lie-Trotter-formula gives

$$e^{-tL} = \lim_{n\to\infty} \left(e^{-\frac{t}{n}\widetilde{L}} e^{-\frac{t}{n}\widetilde{D}}\right)^n.$$

Now, by assumption on b and due to $l(x,y) = -b(x,y)$ we infer that $-\widetilde{L}$ has only non-negative entries. This is then also true of $e^{-\frac{t}{n}\widetilde{L}}$. Also, $e^{-\frac{t}{n}\widetilde{D}}$ has only non-negative entries, as it is a diagonal matrix with some exponential functions on the diagonal. Put together, we infer that e^{-tL} has only non-negative matrix entries. This gives (iii).

(iii)$\Longrightarrow$(ii): From (iii) and the remark we infer $\langle e^{-tL} f, f\rangle \leq \langle e^{-tL}|f|, |f|\rangle$. Moreover, $\langle |f|, |f|\rangle = \langle f, f\rangle$. This gives

$$\langle (e^{-tL} - I)|f|, |f|\rangle \geq \langle (e^{-tL} - I) f, f\rangle.$$

Dividing by $t > 0$ we infer

$$\langle \frac{1}{t} e^{-tL} - I)|f|, |f|\rangle \geq \langle \frac{1}{t}(e^{-tL} - I) f, f\rangle.$$

Taking the limit $t \to 0$ gives then

$$-Q(|f|) = \langle -L|f|, |f|\rangle \geq \langle -Lf, f\rangle = -Q(f).$$

This gives (ii).

(ii)$\Longrightarrow$ (i): Consider $x, y \in X$ with $x \neq y$. Set $f = \delta_x - \delta_y$ (where δ_p is just the characteristic function of p). Then, $|f| = \delta_x + \delta_y$. From (ii) we infer

$$Q(\delta_x + \delta_y) \leq Q(\delta_x - \delta_y).$$

A short computation then gives

$$Q(\delta_x, \delta_y) \leq 0.$$

As $b(x,y) = -l(x,y) = -Q(\delta_x, \delta_y)$ the desired statement (i) follows. □

$\longleftrightarrow$

Having dealt with the first part of the Markov property, we are now going to characterize the second part (provided that the first part holds).

THEOREM. *(2. Beurling / Deny criterion) Let $b : X \times X \longrightarrow [0,\infty)$ be symmetric with vanishing diagonal and $c : X \longrightarrow \mathbb{R}$ be given and let L and Q be the associated operator and form. Then, the following assertions are equivalent:*

(i) $c \geq 0$ *('operator L')*
(ii) $Q(f \wedge 1) \leq Q(f)$ *for all* $f \geq 0$ *('form')*
(iii) $e^{-tL}1 \leq 1$ *('semigroup')*

Remarks.

- Due to the assumption $b \geq 0$, the semigroup e^{-tL} is positivity preserving.
- Given $b \geq 0$, the property $c \geq 0$ in (i) can also be expressed as $\sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$.
- The property in (ii) can be characterized differently:
$$e^{-tL}1 \leq 1 \Longleftrightarrow e^{-tL}f \leq 1 \text{ for all } f \leq 1.$$
($\Longrightarrow$: As e^{-tL} is positivity preserving we have for $f \leq 1$ the following $e^{-tL}f \leq e^{-tL}1 \leq 1$.
$\Longleftarrow$: $f = 1$ possible.)

PROOF. (i)$\Longleftrightarrow$ (iii): Consider $u_t := e^{-tL}1$. Then, $u_0 = 1$ and
$$\frac{d}{dt}u_t = -e^{-tL}L1 = -e^{-tL}c$$
and, in particular,
$$\lim_{t\to 0}\frac{1}{t}(u_t - u_0) = u_0^{'} = -c.$$
Now, if (i) holds then u satisfies $u_0 = 1$ and $u^{'} \leq 0$ (as e^{-tL} is positivity preserving) and (iii) follows. Conversely, if (iii) holds, we infer that $u_0^{'} \leq 0$ and $c \geq 0$ follows.
(i)$\Longrightarrow$ (ii) Recall that $Q(f) = \sum_{x,y\in X} b(x,y)(f(x)-f(y))^2 + \sum_{x\in X} c(x)f(x)^2$. Now, the desired implication follows from a direct computation as for any numbers $a, b \geq 0$ obviously
$$(a\wedge 1)^2 \leq a^2 \;\text{ and}(a\wedge 1 - b\wedge 1)^2 \leq (a-b)^2$$
hold.
(ii)$\Longrightarrow$ (i): Consider $f = 1 + s\delta_x$ with $s > 0$ and $x \in X$ arbitrary. Then, f is nonnegative with $f \wedge 1 = 1$ and (ii) gives
$$Q(1) = Q(f\wedge 1) \leq Q(f) = Q(1) + 2sQ(1,\delta_x) + s^2 Q(\delta_x,\delta_x).$$
A short computation then yields
$$c(x) = Q(1,\delta_x) \geq -\frac{s}{2}Q(\delta_x).$$
As this holds for all $s > 0$, we can take the limit $s \to 0$ and obtain $c(x) \geq 0$. □

We are now going to summarize the preceding two theorems in a characterization of the Markov property. To do so we need a further concept: A map $C : \mathbb{R} \longrightarrow \mathbb{R}$ is called a *normal contraction* if it satisfies
$$C(0) = 0 \;\text{ and }\; |C(a) - C(b)| \leq |a - b|.$$

Note that then $|C(a)| \leq |a|$ must hold for all $a \in \mathbb{R}$.

Examples.

- The map $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}$ is a normal contraction.
- The map $(\cdot)_\pm : \mathbb{R} \longrightarrow [0, \infty), a \mapsto \max\{\pm a, 0\}$ are normal contractions.
- The map $C_I : \mathbb{R} \longrightarrow \mathbb{R}, a \mapsto \max\{0, \min\{a, 1\}\}$ is a normal contraction. (Note also that $C_I(a) = a \wedge 1$ for $a \geq 0$.)

THEOREM. *(Characterization of Markov property) Let $b : X \times X \longrightarrow \mathbb{R}$ be symmetric with vanishing diagonal and $c : X \longrightarrow \mathbb{R}$ be given and let L and Q be the associated operator and form. Then, the following assertions are equivalent:*

(i) $b, c \geq 0$ *('operator L')*
(ii) $Q(C(f)) \leq Q(f)$ *for all f and all normal contractions C ('form')*
(ii)$'$ $Q(C_I(f)) \leq Q(f)$ *for all f ('form')*
(iii) $0 \leq e^{-tL} f \leq 1$ *for all $0 \leq f \leq 1$. ('semigroup)*

Remarks.

- The conditions (ii) and (ii)$'$ 'explain', why $|\cdot|$ and $(\cdot) \wedge 1$ appeared in earlier results.
- Note that by (ii)$'$ compatibility with one special normal contraction suffices to ensure the Markov property.
- Condition (iii) says that the convex set $S := \{f : 0 \leq f \leq 1\}$ is invariant under the semigroup. Note that C_I just acts as the identity on this set.

PROOF. The implication (iii)$\Longrightarrow$ (i) follows immediately from the preceding two theorems. The implication (i)$\Longrightarrow$ (ii) follows by a direct computation from

$$Q(f) = \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f(x)^2$$

and the defining properties of a normal contraction. The implication (ii)$\Longrightarrow$ (ii)$'$ is clear.

It remains to show (ii)$'$ $\Longrightarrow$ (iii). By the previous two theorems it suffices to show

- $f \geq 0$ implies $Q(f \wedge 1) \leq Q(f)$.
- $Q(|f|) \leq Q(f)$.

Here, the first point is immediate from $(ii)'$. As for the second point, it suffices (compare a remark above) to show $Q(f_+, f_-) \leq 0$ for all f. Without loss of generality we will assume $f_+, f_- \leq 1$ (as otherwise we could scale everything). Now, consider $f_s := f_+ - s f_-$ for $s > 0$. Then,

$$Q(f_+) = Q(C_I(f_+ - s f_-)) \stackrel{(ii)'}{\leq} Q(f_+ - s f_-).$$

A short computation then gives

$$Q(f_+, f_-) \leq \frac{s}{2} Q(f_-).$$

Taking the limit $s \to 0$ gives

$$Q(f_+, f_-) \le 0$$

and this finishes the proof. □

DEFINITION. *A symmetric, bilinear form Q on $\mathcal{H}$ is called a Dirichlet form if $Q(C_I(f)) \le Q(f)$ for all $f \in \mathcal{H}$ holds.*

Remark. The preceding theorem just says that Dirichlet forms are in one-to-one correspondence with semigroups having the Markov property.

Remark. Dirichlet forms can be defined and studied in a much more general context viz on σ-finite measure spaces. Then, corresponding statements still hold.

2. Graphs and Dirichlet forms

In this section we discuss how graphs and Dirichlet forms are naturally related.

DEFINITION. *(Graph) Let X be a finite set. A pair (b,c) with $b : X \times X \longrightarrow [0, \infty)$ vanishing on the diagonal and satisfying $b(x, y) = b(y, x)$ for all $x, y \in X$ and $c : X \longrightarrow [0, \infty)$ is called (weighted) graph over X. The elements of X are called vertices and $c(x)$ is called weight of the vertex x. A two-element set $\{x, y\} \subset X$ with $b(x, y) = b(y, x) > 0$ is called (undirected) edge with weight $b(x, y)$. A pair (x, y) with $b(x, y) > 0$ is called directed edge.*

DEFINITION. *(Connectedness) Let (b, c) be a graph over X.*

- *The vertex x is called a neighbor of the vertex y (and y neighbor of x), if $\{x, y\}$ is an edge. We write then $x \sim y$.*
- *A tupel $(x_0, x_1, \ldots, x_{x+1})$ with $x_i \sim x_{i+1}$, $i = 0, \ldots, k$ is called path from x_0 to x_{k+1}.*
- *A graph is called connected if there is a path between any two different points of its underlying set.*

Remark. If a graph is not connected it can be decomposed in connected components. From this point of view we can mostly assume without loss of generality that the graphs in question are connected (see below).

DEFINITION. *Let (b, c) be a graph over X. Then, the operator associated to the graph is given by $L : \mathcal{H} \longrightarrow \mathcal{H}$ with*

$$Lf(x) = \sum b(x, y)(f(x) - f(y)) + c(x)f(x).$$

The form associated to the graph is given by $Q(f, g) = \langle Lf, g\rangle$ and the associated semigroup by $e^{-tL}, t \ge 0$.

Remark. Obviously, graphs over X are in one-to-one correspondence to Dirichletforms over X. Hence, they are in one-to-one correspondence with semigroups having the Markov property.

3. Irreducibility, the ground state and long term behaviour

In this section we study irreducibility, existence of ground states and long term behaviour of the heat equation.

DEFINITION. *An operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ is called positivity improving if $Af(x) > 0$ for all $x \in X$, whenever $f \geq 0$ with $f \neq 0$.*

PROPOSITION. *Let (b, c) be a graph over X with associated operator L. Then, the semigroup e^{-tL} is positivity improving for one (all) $t > 0$ if and only if the graph is connected.*

Proof. $\Longrightarrow$: If (b, c) can be decomposed in two parts, we have $L = L_1 \oplus L_2$ and

$$e^{-tL} = e^{-tL_1} \oplus e^{-tL_2}$$

can not be positivity improving.

$\Longleftarrow$: Let $f \geq 0$ with $f \neq 0$ be given. Set

$$g : [0, \infty) \times X \longrightarrow [0, \infty), \;\; g_t(x) = e^{-tL} f(x).$$

Assume $g_{t_0}(x_0) = 0$ for a $t_0 > 0$ and $x_0 \in X$. Then, $t \mapsto g_t(x_0)$ has a minimum in t_0. Thus,

$$0 = g'_{t_0}(x_0).$$

This implies

$$0 = Lg_{t_0}(x_0) = \sum_y b(x_0, y)(g_{t_0}(x_0) - g_{t_0}(y)) + c(x_0) g_{t_0}(x_0) = -\sum b(x_0, y) g_{t_0}(y).$$

With $g \geq 0$ we conclude $g_{t_0}(y) = 0$ for all $y \sim x_0$. By connectedness of the graph, we obtain inductively $g_{t_0} \equiv 0$. This gives the contradiction $f = e^{tL} g_{t_0} \equiv 0$. □

Remarks.

- Positivity improvingness of the heat semigroup means that heat spreads 'instantanously' over the whole space. In this sense, one can speak about infinite speed of propagation.
- This is a form of 'Minimum principle'.
- (Exercise) Let $P_t = e^{-tL}$ be a positivity preserving semigroup: Then, P_t, $t \geq 0$ is positivity improving if and only if only the trivial subspaces of $\mathcal{H}$ are invariant under the semigroup and multiplication with functions on X.
- Sometimes positivity improving semigroups are called *ergodic*.

We will now turn toward the behaviour at infinity. This will be done in two steps. We first show convergence of the semigroup to the eigenspace of the smallest eigenvalue and then study this eigenspace.

Lemma. *Let (b,c) be a graph over X. Let $P_t = e^{-tL}$ be the associated semigroup and E_1 the projection onto the eigenspace to the smallest eigenvalue λ_1 of L. Then,*

$$\|e^{t\lambda_1}P_t - E_1\| \le e^{-t\alpha}$$

with $\alpha = \lambda_2 - \lambda_1$, where λ_2 is the second smallest eigenvalue of L. In particular,

$$\|P_t - E_1\| \le e^{-t\lambda_2}$$

if $\lambda_1 = 0$.

Proof. Let $L = \sum_{j=1}^k \lambda_j E_j$ with the eigenvalues $\lambda_1 < \lambda_2 < ... < \lambda_k$ and the associated spectral projections on the eigenspaces E_j. As can easily be checked (exercise) we than have

$$e^{-tL} = \sum_{j=1}^k e^{-t\lambda_j} E_j.$$

This gives

$$e^{t\lambda_1}P_t = E_1 + \sum_{j=2}^k e^{-t(\lambda_j-\lambda_1)}E_j.$$

This implies

$$\|e^{t\lambda_1}P_t - E_1\| \le e^{-t(\lambda_2-\lambda_1)},$$

as the E_j are pairwise orthogonal:

$$\begin{aligned}
\|(e^{t\lambda_1}P_t - E_1)f\|^2 &= \sum_{m,n=2}^k e^{-t(\lambda_m-\lambda_1)}e^{-t(\lambda_n-\lambda_1)}\langle E_m f, E_n f\rangle \\
(E_n \text{ pairw. orthogonal}) &= \sum_{m=2}^k e^{-t2(\lambda_m-\lambda_1)}\|E_m f\|^2 \\
&\le e^{-2\alpha t}\sum_{m=1}^n \|E_m f\|^2 \\
(E_m \text{ paarw. orthogonal}) &= e^{-2\alpha t}\|\sum_{m=1}^n E_m f\|^2 \\
&= e^{-2\alpha t}\|f\|^2.
\end{aligned}$$

We conclude that $e^{t\lambda_1}P_t$ converges exponentially fast towards E_1. □

We now turn to investigating E_1.

Theorem. *(Perron-Frobenius) Let (b,c) be a connected graph over X and let L be the associated operator and λ_1 be the smallest eigenvalue of L and E_1 the associated eigenprojection. Then, the eigenspace to λ_1 is one-dimensional and there exists a unique strictly positive eigenfunction e to λ_1 with $E_1 = \langle e,\cdot\rangle e$ i.e. with*

$$E_1 f = \langle e, f\rangle e = (x \mapsto \sum_y e(x)e(y)f(y))$$

for all $f \in \mathcal{H}$.

Proof. We first note the following general fact: A normalized function u is an eigenfunction to λ_1 if and only if

$$\lambda_1 = Q(u).$$

($\Longrightarrow$: This is clear.
$\Longleftarrow$: We have

$$Q(u) = \langle u, Lu\rangle = \langle u, \sum \lambda_j E_j u\rangle = \sum \lambda_j \|E_j u\|^2$$

with $\sum \|E_j u\|^2 = \|u\|^2 = 1$. This gives the desired statement.)

We now show that any eigenfunction to λ_1 is either strict positive or strict negative:
Let f be such a normalized eigenfuntion to λ_1. Then,

$$\lambda_1 = Q(f) \geq Q(|f|) \geq \lambda_1.$$

Here, we used that Q is a Dirichlet form in the middle step. This implies

$$\lambda_1 = Q(|f|).$$

As $|f|$ is normalized as well, we infer that $|f|$ is again an eigenfunction to λ_1.
We now decompose f in its positive part f_+ and its negative part f_-. With f and $|f|$, we then have that

$$f_+ = \frac{1}{2}(f + |f|), \;\; f_- = \frac{1}{2}(|f| - f_+)$$

are again eigenfunctions to λ_1 (or vanish identically). Assume w.l.o.g. $f_+ \neq 0$. As P_t is positivity improving we infer

$$0 < P_1 f_+ = e^{-\lambda_1} f_+.$$

This gives

$$f_+ > 0 \;\text{ and }\; f_- = 0.$$

These considerations show that any eigenfunction to λ_1 has a strict sign.

We conclude that the eigenspace to λ_1 one-dimensional ist (as eigenfunctions with a strict sign can not be orthogonal to each other).

As the eigenspace to λ_1 is one-dimensional we obtain

$$E_1 f = \langle e, f\rangle e$$

for any normalized eigenfunction e. Hence, any normalized strict positive e has the desired properties (and is uniquely determined by them). □

Remark. (Exercise) Let (b, c) be a graph over X with $c \equiv 0$. Let $P_t = e^{-tL}$ be the associated semigroup.

- If the graph is conneced, then $\lambda_1 = 0$ and the eigenspace consists exactly of the constant functions. (Hint: Obviously the constant functions are eigenfunctions to 0. Conversely, let f be an eigenfunction to 0 and consider now $0 = Q(f) = \frac{1}{2}\sum_{x,y} b(x,y)(f(x) - f(y))^2 ...$)
- If the graph is not connected, then $\lambda_1 = 0$ and the eigenspace consists exactly of those functions which are constant on each connected component.

COROLLARY. *In the situation of the previous theorem, $\lambda_1 = 0$ if and only if $c \equiv 0$. If $c \neq 0$, then $\lambda_1 > 0$.*

Proof. (Notation as in the theorem and its proof.)

$$\lambda_1 = Q(e,e) = \sum_{x,y} b(x,y)(e(x) - e(y))^2 + \sum_{x \in X} c(x)e(x)^2.$$

This easily gives the statement. □

DEFINITION. *Let (b,c) be a connected graph over X with associated operator L. Then, the smallest eigenvalues λ_1 of L is called the ground state energy and the normalized positive eigenfunction e to λ_1 is called the ground state.*

We can now turn to the main result of this section.

THEOREM. *Let (b,c) be a connected graph over X and $P_t = e^{-tL}$, $t \geq 0$, the associated semigroup with ground state energy λ_1 and ground state e. Then, the following holds:*

- $\lim_{t \to \infty} \frac{\ln P_t(x,y)}{t} = -\lambda_1$ *for all $x, y \in X$.*
- $|e^{t\lambda_1} P_t(x,y) - e(x)e(y)| \leq e^{-t\alpha}$ *for all $x, y \in X$, where $\alpha = \lambda_2 - \lambda_1 > 0$.*

Proof. *Second statement:* The previous theorem gives $E_1(x,y) = e(x)e(y)$. Now, the second statement follows from the first lemma of this section.

First statement: By the already proven second statement we infer

$$e(x)e(y) - e^{-t\alpha} \leq e^{t\lambda_1} P_t(x,y) \leq e(x)e(y) + e^{-t\alpha}.$$

As e is strict positive by the previous theorem, the desired statement follows after taking logarithms, dividing by t and taking the limit $t \to \infty$. □

COROLLARY. *Let (b,c) be a connected graph over X with $c \equiv 0$. Let L the associated operator and $P_t = e^{-tL}$, $t \geq 0$, the associated semigroup. Then,*

$$\left| P_t(x,y) - \frac{1}{N} \right| \leq e^{-t\lambda_2}.$$

Proof. By $c \equiv 0$ the ground state energy is given by 0 and a (the) normalized strictly positive eigenfunction is given by the constant function with value $1/\sqrt{N}$. Now, the statement follows from the previous theorem. □

Remark. In the theorem (and its corollary) one obtains expoential convergence towards the ground state. The rate depends on the distance between the two first eigenvalues, i.e. the so-called spectral gap. This motivates study of this spectral gap.

4. Connection to Markov processes

Markov processes play an important role in various branches of e.g. physics, mathematics, ecconomy, biology. They lead to semigroups with the Markov property. This is a main motivation in the study of such semigroups. Here, we are going to give an interpretation of the Markov property in terms of such processes. To do so, we will discuss Markov processes (without actually constructing or definiing them rigorously).

Sketch of what is done in this section. $(a,w) \overset{Stochastics}{---\!>}$ Markov process $\overset{s.below}{---\!>}$ Markovian semigroup $(P_t) \overset{Theorem}{---\!>} L = \frac{d}{dt}P_t ---> (a,w)$.

A Markov process in continuous time on the discrete set X models a particle jumping on $X \cup \{\infty\}$ according to spefic rules discussed next. Each $x \in X$ has assigned two quantities viz

- a number $a_x > 0$ and
- a function $w_x : X \setminus \{x\} \longrightarrow [0,\infty)$ with $\sum w_x(y) \leq 1$.

The particle now moves according to the following procedure:

If the particle is at ∞ nothing happen s. Otherwise, if the particle has just jumped to $x \in X$ it carries out two independent random experiments:

- It randomly choses a waiting time according to some exponential distribution with parameter a_x (i.e. the probability to be at time t still in x is given by $= e^{-ta_x}$).
- It randomly choses $y \in X$ with $y \neq x$ as aim for the next jump with probability $w_x(y)$ and the aim ∞ with probability $1 - \sum_z w_x(z)$.

After this the particle waits the waiting time and then jumps (in no time) to the aim. Now, the whole procedure starts again.

Important. In this procedure, the waiting time and the aim only depend on the present position of the particle (and not on the way how this position was reached).

Remark. There are processes in which the waiting time is fixed (e.g. to 1). These are called Markovian processes in discrete time or Markov chains. Here, we deal with Markov processes in continous time.

If there exists at all a stochastic process giving a precise version for the above procedure, then the probabilities

$P_t(x,y) =$ probability to be at time t in y after start in in x at time 0

should satisfy:

- $P_tP_s = P_{t+s}$ i.e.
$$P_{t+s}(x,y) = \sum P_t(x,z)P_s(z,y).$$
('Independence of history')
- $P_t(x,y) = P_t(y,x)$. ('Reversibility')
- $\lim_{t\to 0} P_t\delta_x = \delta_x$. ('Continuity')
- $P_t(x,y) \geq 0$ all x,y and $\sum_y P_t(x,y) \leq 1$. ('Probabilities')

It can be proven that any P_t satisfying the first three properties must have the form e^{-tL} with a selfadjoint operator L. The last property the means that L has the form discussed in the previous sections. Thus, the $P_t = e^{-tL}$ are a semigroup with the Markov property.

We are now going to investigate how the quantities (a,w) specifiying the Markov process are related to the matrix elements of L.

Note that $P_t = e^{-tL}$ is differentiable with

$$-l(x,y) = \lim_{t\to 0} \frac{1}{t}(P_t(x,y) - 1_{x,y})$$

for all $x, y \in X$. This can be interpreted as follows:

Case 1. $x = y$. For really small $t > 0$ the occurrence of two jumps is rather unlikely. Thus, we have

$$P_t(x,x) \sim \text{ Probability to not have jumped at time } t \ = e^{-a_x t},$$

which gives

$$-l(x,x) = \lim_{t\to 0} \frac{1}{t}(e^{-a_x t} - 1) = -a_x.$$

Put differently

$$a_x = l(x,x) = \sum_z b(x,z) + c(x).$$

Case 2. $x \neq y$. For really small $t > 0$ the occurrence of two jumps is rather unlikely. Thus, we have

$$\begin{aligned} P_t(x,y) &\sim \ \text{Probability of up to time } t \text{ having jumped only once with aim } y \\ &= \text{Probability (jumping time} \le t \text{) Probability (aim of jump is } y) \\ &= (1 - e^{-a_x t}) w_x(y). \end{aligned}$$

Here, we used in the middle step that waiting time and aim of jump are independent of each other. This gives

$$b(x,y) = -l(x,y) = \lim_{t\to 0} \frac{1}{t}(1 - e^{-a_x t}) w_x(y) = a_x w_x(y).$$

Put differently

$$w_x(y) = \frac{b(x,y)}{a_x} = \frac{b(x,y)}{\sum_z b(x,z) + c(x)} = \frac{-l(x,y)}{l(x,x)}.$$

In this approach

- the diagonal elements of L control the distribution of the waiting time,
- the off-diagonal elements of L control the probability distribution for the aims of the jump.

Remark. If $c_1, c_2 \equiv 0$ and $b_1, b_2 : X \times X \longrightarrow [0, \infty)$ are symmetric with vanishing diagonal with

$$\frac{b_1(x,y)}{\sum_z b_1(x,z)} = \frac{b_2(x,y)}{\sum_z b_2(x,z)}$$

for all $x, y \in X$, then the associated semigroups encode the same statistics of the jump aims. However, the time used to go along the 'orbits' can be very different.

$\longleftrightarrow$

Remark. Many basic results on Markov semigroups can be very easily understood in terms of the underlying process. We give two examples. In both cases, a connected graph (b, c) over X is given.

- The semigroup is positivity improving if and only if the graph is connected, ('Proof' in the Markov process picture: The term $e^{-tL}(x,y)$ is the positive (!) probability of the particle managing to get from x to y in the time t.)
- (Exercise) $c \neq 0$ if and only if $e^{-tL}1 \neq 1$. ('Proof' in the Markov process picture: By general principles we know $e^{-tL}1 \leq 1$. Now, strict equality will hold if and only if the system looses mass. This can only happen by the particle jumping out. This in turn is equivalent to having on $x \in X$ with $c(x) > 0$.)

CHAPTER 2

Electrostatics and the Poisson equation on finite sets

Introduction: Electrostatics and Poisson equation

In this section we study the equation

$$(L+\alpha)u = \varrho \ \text{ on } S \subset X \ \ u = g \ \text{ on } X \setminus S$$

with X and L as above. Here, $\alpha \geq 0$ and ϱ, g and $S \subset X$ are given and u is the unkown solution. The interpretation in terms of electrostatics of this equation is as follows: ϱ is the given charge distribution, g is the potential at the boundary. We look for the arising electrostatical potential induced by the charge under the restriction that it takes the prescribed value on the boundary. This potential is the solution of the above equation (for $\alpha = 0$). Most important special cases concern the situation $\alpha = 0$ and/or $S = X$, $g = 0$. This leads to harmonic functions and to the Dirichlet problem.

1. Networks and harmonic functions

We start by describing the situation and fixing some notation. In this section we consider a graph $(b, 0)$ over $X = \{1, \ldots, N\}$ and write it shortly as (X, b) and call this a *network*. A pair $(x, y) \in X \times X$ with $b(x, y) > 0$ is then called a *directed edge*. For a directed edge $e = (x, y)$ we define the source by $x =: s(e)$ and the range by $y =: r(e)$ and the reverse edge $\overline{e}$ by $\overline{e} = (y, x)$. The set of all directed edges is then denoted by $E = E(X, b)$. The function

$$w : E \longrightarrow \mathbb{R}, w((x,y)) = \frac{1}{b(x,y)}$$

is called *Resistence* and b is called the *conductance*.
A tuple $(e_1, \ldots, e_n)$ of edges is called *cycle*, if

$$r(e_j) = s(e_{j+1}), j = 1, \ldots, n,$$

where we set $e_{n+1} = e_1$.
A map

$$\varphi : E \longrightarrow \mathbb{R}$$

is called *flow*, if $\varphi(e) = -\varphi(\overline{e})$. Die *Energy* of a flow φ is defined by

$$\mathcal{E}(\varphi) = \frac{1}{2} \sum_{e \in E} \varphi(e)^2 w(e).$$

Interpretation. We deal with flows and functions on networks. In this context it is useful to have in mind a static (!) situation of currents in a system of wires or water in a system of pipes:

- Functions on the vertices correspond to potentials i.e. (differences in) voltage or pressure on the knots.
- Flows correspond to electrical currents or water flows.
- Resistance corresponds to electrical resistance or thickness of tubes.

Ohms law is valid saying

$$U/I = R \text{ i.e. } \frac{\text{potential difference}}{\text{flow}} = \text{resistance}.$$

The corresponding *energy* is given by

$$\frac{1}{2}UI = \frac{1}{2}\frac{U^2}{R} = \frac{1}{2}I^2R.$$

We will come back to this interpretation from time to time in the sequel.

DEFINITION. *Let (X, b) be a network and $\varphi : E \longrightarrow \mathbb{R}$ a flow. Then, φ is said to satisfy the Kirchoff cycle rule (KCR) if*

$$\sum_{j=1}^{n} \varphi(e)w(e) = 0$$

for any cycle $(e_1, \ldots, e_n)$.

Example. Let $f : X \longrightarrow \mathbb{R}$ be a function. Then, $\Psi_f : E \longrightarrow \mathbb{R}$ defined via

$$\Psi_f(e) = (f(r(e)) - f(s(e)))b(s(e), r(e)) = \frac{f(r(e)) - f(s(e))}{w(e)}$$

is a flow satisfying Kirchoff cycle rule (KCR). It is called the flow induced by f. Clearly,

$$\Psi_{f+\lambda g} = \Psi_f + \lambda \Psi_g.$$

In fact, the preceding example is not just one example but rather **the** example of a flow satisfying Kirchhoff cycle rule. This is the content of the next proposition.

PROPOSITION. *Let (X, b) be a network and φ a flow on it. Then, the following assertions are equivalent:*

(i) *The flow φ satisfies the Kirchhoff cycle rule.*
(ii) *There exists an $f : X \longrightarrow \mathbb{R}$ with $\varphi = \Psi_f$.*

In this case $\Psi_{f_1} = \Psi_{f_2}$ if and only if $f_1 - f_2$ is constant on each connected component.

Remark. The last statement is known from physics as the arbitraryness in fixing the zero of the potential.

Proof. (ii)$\Longrightarrow$ (i): This is already known from the example.

(i)$\Longrightarrow$ (ii): Without loss of generality let (X, b) be connected (otherwise we can argue on each connected component separately). Fix $o \in X$ and define $f(o) = 0$. Chose now for any $x \in X$ a path $(x_0, \ldots, x_n)$ in X with $x_0 = o$ and $x_n = x$ (this is possible as we have connectedness) and define

$$f(x) := \sum_{j=0}^{n-1} \varphi(x_j, x_{j+1})w(x_j, x_{j+1}).$$

Due to Kirchhoff cycle rule this is well defined. By construction we have for x, y with $x \sim y$ then

$$f(y) = f(x) + \varphi(x,y)w(x,y)$$

i.e.

$$\frac{(f(y) - f(x))}{w(x,y)} = \varphi(x,y).$$

This gives (ii).

As for the last statement: Assume again without loss of generality that (X, b) is connected and let $\Psi_{f_1} = \Psi_{f_2}$. Thus,

$$0 = \Psi_{f_1 - f_2}.$$

With $f = f_1 - f_2$ we infer

$$0 = \frac{f(r(e)) - f(s(e))}{w(e)}$$

for any edge e. As the graph is connected we conclude that $f = const$ and this is the desired statement. □

The proposition says that functions on the vertices are equivalent to flows (on the edges) satisfying Kirchhoff cycle rule. Accordingly, it is possible to 'translate' statements from the world of functions into the world of flows and vice versa. This will be studied next.

PROPOSITION. *Let (X, b) be a network with associated form Q. Let φ be a flow on (X, b) with $\varphi = \Psi_f$ for an $f : X \longrightarrow \mathbb{R}$. Then,*

$$\mathcal{E}(\varphi) = Q(f)$$

holds.

Remark. This is a version of $\frac{1}{2}\frac{U^2}{R} = \frac{1}{2}I^2 R$.

Proof. This follows by a direct computation:

$$\begin{aligned}
\mathcal{E}(\varphi) &= \frac{1}{2}\sum_{e \in E} \varphi(e)^2 w(e) \\
&= \frac{1}{2}\sum_{(x,y) \in X} \varphi(x,y)^2 \frac{1}{b(x,y)} \\
&= \frac{1}{2}\sum_{x,y \in X} b(x,y)(f(x) - f(y))^2 \\
&= Q(f).
\end{aligned}$$

Here, we used $\varphi = \Psi_f$ d.h. $\varphi(x,y) = b(x,y)(f(y) - f(x))$ in the previous to the last line. □

We now turn to a second important property that a flow may satisfy.

Definition. *Let (X, b) be a network. Then, the flow φ satisfies Kirchhoff vertex rule (KVR) in $x \in X$, if*

$$\sum_{e:x=r(e)} \varphi(e) = 0.$$

A flow satisfies Kirchhoff vertex rule if it satisfies it in every vertex.

Remark. If the flow φ satisfies (KVR) in $x \in X$, we have

$$0 = \sum_{e:x=s(e)} \varphi(e)$$

(and vice versa). In fact, for any decomposition $E_1 \cup E_2 = E_x = \{e : r(e) = x\}$ we then have

$$0 = \sum_{e \in E_1} \varphi(e) - \sum_{e \in E_2} \varphi(\overline{e}).$$

Remark. By the laws of electrostatics the current in a network of wires satisfies both Kirchhoff cycle rule and Kirchhoff vertex rule. Similarly, both Kirchhoff rules are 'obvious' for (static) flow of water in a network of pipes.

We now give an interpretation of Kirchhoff vertex rule for flows (coming from functions).

Definition. *Let (X, b) be a network with associated operator L. Then, $f : X \longrightarrow \mathbb{R}$ is called harmonic on $A \subset X$, if $Lf = 0$ on A. If f is harmonic on $A = X$, it is called harmonic.*

It is not hard to characterize under which conditions $\varphi = \Psi_f$ satisfies KVR.

Lemma. *Let (X, b) be a network and $\varphi = \Psi_f$. Then, the following assertions are equivalent for $x \in X$:*

(i) *The flow φ satisfies Kirchhoff vertex rule in x.*
(ii) *The equality $\sum_y b(x, y)(f(x) - f(y)) = 0$ holds (i.e. f is harmonic in $\{x\}$).*

In particular, $\varphi = \Psi_f$ satisfies Kirchhoff vertex rule if and only if $Lf = 0$ holds.

Proof. Due to

$$\varphi(x, y) = (f(y) - f(x))b(y, x)$$

this is immediate from the definitions. □

2. Dirichletproblem and condensator principle

In this section we study the following 'boundary value' problem, also known as Dirichlet problem: Let (b, c) be a graph over X and let a subset $B \subset X$ ('the boundary ') be given and g a function on B. Now, we look for a function u on X satisfying

- $Lu = 0$ on $A := X \setminus B$ ('u is harmonic')
- und $u = g$ on B ('u takes the value g on the boundary')

Remark. One of the basic problems of electrostatic of networks consists in finding the flow generated by a voltage given at certain points. By the laws of electrostatics such a flow will satisfy Kirchoff cycle rule. Thus, it is induced from a function. This function must satisfy Kirchhoff vertex rule in all points, where there is no voltage given. Thus, we are lead to the above problem.

Theorem. *Let (b, c) be a connected graph over X. Let $B \subset X$ with $B \neq \emptyset$ $A := X \setminus B$ and $g : B :\longrightarrow \mathbb{R}$ be given and define $\mathcal{A}_g := \{h \in \mathcal{H} : h = g \text{ on } B\}$. Then, the Dirichlet problem (DP)*

- $Lu = 0$ *on* A
- $u = g$ *on* B

has a unique solution and for $f \in \mathcal{A}_g$ the following assertions are equivalent:

(i) $Q(f) = \min\{Q(h) : h \in \mathcal{A}_g\}$.
(ii) *The function f solves (DP).*

In particular, the minimizer in (i) is unique. Moreover, if $0 \leq g \leq 1$ then $0 \leq f \leq 1$ holds.

Remarks.

- The theorem says that the solution minimizes the energy (as is sensible for a solution to a physical problem).
- For $B = \emptyset$, the corresponding statement is wrong in general. For example $Lu = 0$ does not have a unique solution for $c = 0$.

Proof. We will show a series of claims which will prove the theorem (and a bit more).

The solution of (DP) exists and is unique. We transform the problem in an equivalent problem that we solve.

Let f be a solution of $0 = Lf$ on A with $f = g$ on B. For any $x \in A$ we then have

$$\begin{aligned}
0 = Lf(x) &= \sum_{y \in X} b(x,y)(f(x) - f(y)) + c(x)f(x) \\
&= \sum_{y \in A} b(x,y)(f(x) - f(y)) + \sum_{y \in B} b(x,y)f(x) - \sum_{y \in B} b(x,y)f(y) + c(x)f(x) \\
&= \sum_{y \in A} b(x,y)(f(x) - f(y)) + f(x)\sum_{y \in B} b(x,y) - \sum_{y \in B} b(x,y)g(y) + c(x)f(x) \\
&= \sum_{y \in A} b(x,y)(f(x) - f(y)) + C_x f(x) - h(x)
\end{aligned}$$

with

$$C_x := c(x) + \sum_{y \in B} b(x,y), \text{ and } h(x) = \sum_{y \in B} b(x,y)g(y),$$

which do not depend on f! Invoking the operator $\widetilde{L}_A$ associated to the graph (b_A, C) over A and the restriction $\widetilde{f}$ of f on A, we obtain

(P) $\widetilde{L}_A \widetilde{f} = h$.

Now, if f is a solution of (DP), then $\widetilde{f}$ solves (P) (as shown by the above calculation). Conversely, any solution $\widetilde{f}$ of (P) becomes a solution f of (DP) after being extended by g on B. This gives:

$$f \text{ solves (DP)} \Longleftrightarrow \widetilde{f} \text{ solves (P)}.$$

It suffices therefore to show that (P) has a unique solution. To do so, it suffices to show that 0 does not belong to the spectrum of $\widetilde{L}_A$.

By construction $\widetilde{L}_A$ is induced by the graph (b_A, C) over A. Thus, it remains to show that C does not vanish on any connected component (w.r.t. b_A) of A (as then the infimum of the spectrum of $\widetilde{L}_A$ is strictly positive as discussed above). Let Z be such a connected component. It suffices to find an $y \in B$ with $b(x,y) > 0$ for an $x \in Z$ (compare definition of C). Chose now first arbitrary $y^{'}$ in B and $o \in Z$. As the graph is connected there exists a path $(x_0, x_1, \ldots, x_n)$ in (X, b) with $x_0 = o$ and $x_n = y^{'}$. Let j be the smallest index, such that x_j does not belong to Z. Then $y := x_j$ belongs to B (as otherwise it would belond to Z). Thus, $y = x_j$ has the desired properties.

Any f with $Q(f) = \min\{Q(h) : h \in \mathcal{A}_g\}$ solves (DP). Let φ be an arbitrary function supported on A. Then, $f + \lambda\varphi$ belongs to $\mathcal{A}_g$ for all $\lambda \in \mathbb{R}$. Thus, the function

$$\lambda \mapsto Q(f + \lambda\varphi) = Q(f) + 2\lambda Q(f, \varphi) + \lambda^2 Q(\varphi)$$

possess a minimum for $\lambda = 0$. Taking the derivative yields

$$0 = Q(f, \varphi) = \langle Lf, \varphi \rangle.$$

As φ was arbitrary (with support in A) we conclude $Lf = 0$ on A.

There exists an minimizer of Q on $\mathcal{A}_g$. Let (f_n) be a sequence in $\mathcal{A}_g$ with

$$Q(f_n) \to \inf\{Q(h) : h \in \mathcal{A}_g\}.$$

Then, $(Q(f_n))_n$ is bounded. Let o be an arbitrary point in B. Then, $f_n(o) = g(o) = \mathit{constant}$. It turns out (! see below) that the boundedness of the $Q(f_n)$ together with the boundedness of the $(f_n(o))$ implies that $(f_n(x))_n$ is bounded for any $x \in X$. By chosing a suitable subsequence we can then without of loss of generality assume that (f_n) converges pointwise to a function f. This f must then belong again to $\mathcal{A}_g$ and

$$Q(f) \le \liminf_{n\to\infty} Q(f_n) = \inf\{Q(h) : h \in \mathcal{A}_g\}$$

holds. Thus, f is a minimizer.

(! It remains to show the desired bounedness of $(f_n(x))$ for $x \in X$: Let $x \in X$ be given and $\gamma = (x_0, \ldots, x_n)$ with $x_0 = o$ and $x_n = x$ a path from o

to x with pairwise different x_j. Then, for any function u we have

$$\begin{aligned}|u(x)-u(o)| &\leq \sum_{j=1}^{n}|u(x_j)-u(x_{j+1})|\\ &= \sum_{j=1}^{n}|u(x_j)-u(x_{j+1})|b(x_j,x_{j+1})^{1/2}\cdot\frac{1}{b(x_j,x_{j+1})^{1/2}}\\ &\leq \left(\sum_{j=1}^{n}|u(x_j)-u(x_{j+1})|^2 b(x_j,x_{j+1})\right)^{1/2}\left(\sum_{j=1}^{n}b(x_j,x_{j+1})^{-1}\right)^{1/2}\\ &\leq Q(u)^{1/2}C(\gamma)\end{aligned}$$

with $C(\gamma)=\left(\sum_{j=1}^{n}b(x_j,x_{j+1})^{-1}\right)^{1/2}$. This gives the desired statement).

$0\leq g\leq 1$ *implies* $0\leq f\leq 1$. With f also $C_I(f)$ belongs to $\mathcal{A}_g$ and is also a minimizer of Q (as Q is a Dirichlet form). The already proven uniqueness then gives $f=C_I f$ and this means $0\leq f\leq 1$.

Taken together the preceding statements prove the theorem. □

Remark (Various metrics on graphs). On the graph (b,c) over X one can introduce the metrics

$$d_1(x,y):=\inf\{\sum_{j=0}^{n-1}\frac{1}{b(x_j,x_{j+1})}:\gamma=(x_0,\ldots,x_n)\ \text{path from } x \text{ to } y\}$$

and

$$d_2(x,y):=\inf\{\sum_{j=0}^{n-1}\frac{1}{b(x_j,x_{j+1})^{1/2}}:\gamma=(x_0,\ldots,x_n)\ \text{path from } x \text{ to } y\}.$$

Then, obviously the inequality

$$d_1\leq d_2^2$$

holds. Moreover, the last part of the proof of the previous theorem shows

$$|u(x)-u(y)|\leq Q(u)^{1/2}d_1(x,y)^{1/2}$$

and we infer

$$|u(x)-u(y)|\leq Q(u)^{1/2}d_2(x,y).$$

Thus, any function u is d_2 Lippschitz continuous and d_1 Hoelder continuous (with exponent 1/2), where the constants are given by the form applied to u. This type of inequality plays a role when dealing with infinite graphs as it allows one to extend functions with finite values of Q to suitable completions of the graph.

A consequence of the theorem is the existence of the so called effective resistance W_{eff}.

COROLLARY. *(Existence of effective resistance) Let (X,b) be a network and $s,t\in X$ with $s\neq t$ be given. Then, there exists a unique $f=f_{s,t}$ with $f(s)=0$, $f(t)=1$ and $Lf=0$ on $X\setminus\{s,t\}$. This f is a minimizer of Q on $\mathcal{A}_{s,t}:=\{h:h(s)=0,h(t)=1\}$.*

Remark (Effective Resistance and resistance metric). (a) If one puts a normalized voltage between s and t then the effective resistance $W_{\text{eff}}(s,t)$ between s and t satisfies

$$W_{\text{eff}}(s,t) = \frac{1}{Q(f_{s,t})}.$$

This quantity plays an important role in the theory of networks.
(The reason for this formula is that the energy E of the system is given by - as we have seen above - both E $= UI = U^2/R$ and $E = Q(f)$. Now, with $f = f_{s,t}$ and $U = 1 = 1 - 0$ the formula follows.)
(b) (Exercise) Under the condition that $c = 0$, the effective resistance can also be expressed by the following remarkable formula

$$W_{\text{eff}}(s,t) = \max\{|f(s) - f(t)|^2 : Q(f) \le 1\}.$$

(Hint: Let $x, y \in X$ with $x \neq y$ be given. Using $Q(f) = Q(f + c1)$ for any $c \in \mathbb{R}$ and $Q(f) = Q(-f)$ it is possible to show (how?) that

$$\begin{aligned} \min\{Q(f) : f(x) = 0, f(y) = 1\} &= \min\{Q(f) : |f(x) - f(y)| = 1\} \\ &= \min\{\frac{Q(f)}{|f(x) - f(y)|^2} : f(x) \neq f(y)\}. \end{aligned}$$

This gives (why?)

$$W_{\text{eff}}(x,y) = \max\{\frac{|f(x) - f(y)|^2}{Q(f)} : f(x) \neq f(y)\} = \max\{|f(x)-f(y)|^2 : Q(f) \le 1\}.$$

This shows the statement.)
(c) A large part of the relevance of the effective resistance comes from the fact that (exercise)

$$\varrho(s,t) := W_{\text{eff}}^{1/2}(s,t) \text{ for } s \neq t \text{ and } \varrho(s,t) = 0 \quad \text{for } s = t$$

defines a metric on the graph, which is particularly adapted to the graph. This metric is called the *resistance metric* on the graph.

As a consequence of the previous theorem we obtain the condensator principle.

COROLLARY. *(Condensator principle) Let (b,c) be a graph over X with associated operator L. Let $E, F \subset X$ be subets of X with $E \cap F = \emptyset$ and $E \cup F \neq \emptyset$. Then, the condensator problem (CP)*

- *$u \equiv 1$ and $Lu \ge 0$ on E,*
- *$u \equiv 0$ and $Lu \le 0$ on F,*
- *$Lu = 0$ otherwise,*

has a unique solution. This solution is given by the minimizer of Q on $\mathcal{A} = \{h : h \ge 1$ on E, $h \le 0$ on $F\}$.

Proof. By the preceding theorem the problem
- $Lu = 0$ auf $X \setminus (E \cup F)$,
- $u \equiv 1$ on E,
- $u \equiv 0$ on F
possesses a unique solution and this solution satisfies $0 \le u \le 1$.
Hence, this solution u satisfies on E

$$Lu(x)=\sum_{y\in X}b(x,y)(u(x)-u(y))+c(x)u(x)=\sum_{y\in X}b(x,y)(1-u(y))+c(x)\geq 0$$

and on F

$$Lu(x)=\sum_{y\in X}b(x,y)(u(x)-u(y))+c(x)u(y)=\sum_{y\in X}b(x,y)(0-u(y))\leq 0.$$

It is therefore a solution of the condensator problem. It must be unique (as already the solution of the Dirichlet problem is unique).

Moreover, this solution is the unique minimizer of Q on

$$\mathcal{A}_g:=\{h: h=1 \text{ on } E,\ h=0 \text{ on } F\}.$$

Now, obviously $\mathcal{A}_g\subset\mathcal{A}$. As Q is a Dirichletform and $C_I\mathcal{A}=\mathcal{A}_g$ the desired statement on the minimizer follows. □

We are now ready to prove the following characterization of Dirichlet forms in electrostatics.

Theorem. *(Characterization of Dirichletforms in electrostatics) Let Q be a symmetric form on $\mathcal{H}$ with associated operator L. Then, the following assertions are equivalent:*

(i) *Q is a Dirichlet form.*
(ii) *Any Dirichlet problem (DP) $Lu=0$ on A, $u=g$ on B (with $A\cup B=X$ and $A\cap B=\emptyset$) has a unique solution on X and this solution satisfies $0\leq f\leq 1$ if $0\leq g\leq 1$.*
(iii) *Any condensator problem (CP) on X has a unique solution.*

Proof. (i)$\Longrightarrow$ (ii): This is shown in the previous theorem.

(ii)$\Longrightarrow$ (iii): This is clear (compare also proof of previous corollary).

(iii)$\Longrightarrow$ (i): Let $x\in X$ be arbitrary. With $E=\{x\}$ and $F=X\setminus\{x\}$ i.e. $u=\delta_x$ we obtain from (CP) immediately $Lu\leq 0$ on F and hence for all $y\neq x$

$$0\geq L\delta_x(y)=l(x,y).$$

With $E=X$ i.e. $u\equiv 1$ we obtain from (CP) immediately $Lu\geq 0$ and hence

$$0\leq\sum_y L(x,y).$$

This shows that Q is a Dirichlet form. □

Remark. The theorem says that Dirichlet forms are the 'right' objects to do electrostatics. It goes back to the work of Beurling / Deny '58.

3. Remark on heat equation and electrostatics

. It is truly remarkable that the same mathematical structure (viz Dirichlet forms) appear prominently in both the theory of heat equation and electrostatics. This is not only true in the discrete setting considered in the previous two chapters but also in the continuum setting. There, the operator L is replaced by the Laplacian $-\Delta$. Here, we will briefly discuss the mathematics behind this phenomenon.

Our considerations show that the heat equation is about the semigroup e^{tL}, $t \geq 0$. Electrostatics on the other hand deals Poisson equation and Dirichlet Problem. As seen in our discussion of the Dirichlet Problem this finally yields equations of the form

$$(L + \alpha)u = g$$

(where it is sometimes necessary to modify the underlying graph). In this sense, electrostatics yields the resolvents $(L + \alpha)^{-1}$, $\alpha > 0$.

Mathematically, semigroups and resolvents are intimately related. In fact, one can be obtained from the other. The corresponding formulae are the following (exercise):

$$(L + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} dt$$

for $\alpha > 0$ and

$$e^{-tL} = \lim_{n \to \infty} \Big(\frac{n}{t} (\frac{n}{t} + L)^{-1} \Big)^n$$

for all $t > 0$.

CHAPTER 3

Cheeger inequality and lower bounds

In this section we consider again a Dirichlet form on a finite set and present (one variant of) a Cheeger inequality and use it to obtain lower bounds for the first and second eigenvalue.

1. Co-Area formulae

The Co-area formulae are an important tool in dealing with lower bounds on forms.

We consider the follwing situation: Let X be finite and $b : X \times X \longrightarrow \mathbb{R}$ symmetric with $b(x,x) = 0$ for $x \in X$.
For $K \subset X$ we define the *surface* or *strength of connection* between K and $X \setminus K$ (in terms of b) via

$$S_K := \sum_{x \in K, y \notin K} b(x,y).$$

Note that $S_K = S_{X \setminus K}$ as b is symmetric.
For $f : X \longrightarrow \mathbb{R}$ and $t \in \mathbb{R}$ we set

$$K_t := K_t^f := \{x \in X : f(x) \geq t\}.$$

Thus, $K_t = X$ for $t \leq \min f$ and $K_t = \emptyset$ for $t > \max f$. Let now $t_0 < t_1 < \ldots < t_k$ the different values of f and set

$$\widetilde{K}_i := K_{t_i}.$$

Then, obviously

$$K_t = \widetilde{K}_i \text{ for } t_{i-1} < t \leq t_i.$$

We finally set for $a, b, t \in \mathbb{R}$

$$1_{(a,b]}(t) \equiv \begin{cases} 1 & : \quad a < t \leq b \\ 0 & : \quad sonst. \end{cases}$$

Note that this means in particular, that $1_{(a,b]}(t) \equiv 0$, whenever $b \leq a$.

Theorem. *(Co-Area-formula) Let $b : X \times X \longrightarrow \mathbb{R}$ symmetric with $b(x,x) = 0$ for $x \in X$ and $f : X \longrightarrow \mathbb{R}$ be given. Then,*

$$\frac{1}{2} \sum_{x,y \in X} b(x,y)|f(x) - f(y)| = \int_{-\infty}^{\infty} S_{K_t} dt = \sum_{i=1}^{k} (t_i - t_{i-1}) S_{K_{t_i}}.$$

Remark.

- Note that $S_{K_t} = \sum_{(x,y): f(y) < t \leq f(x)} b(x,y)$.

- The last equality is more or less clear. The first equality can be seen as a form of Fubinig / Cavaliere argument: To do this we can draw a coordinate system in the plane. On the x axis we put an interval of length $b(x, y)$ for any edge. On the y-axis we put the values of f. The desired sum is then the sum of the areas of the corresponding rectangles. This can be obtained by both integration over the x-axis and integration over the y-axis. This gives the desired formula.

Proof. We call the relevant terms in the chain of equalities T_1, T_2, T_3 (in order of appearance).

The equality $T_1 = T_2$ holds. It is easy to see that

$$S_{K_t} = \sum_{x \in K_t, y \notin K_t} b(x,y) = \frac{1}{2}\sum_{x,y}(1_{(f(x),f(y)]}(t) + 1_{(f(y),f(x)]}(t))b(x,y).$$

This implies

$$\begin{aligned}\int_{-\infty}^{\infty} S_{K_t} dt &= \frac{1}{2}\sum_{x,y} b(x,y)\int_{-\infty}^{\infty}(1_{(f(x),f(y)]}(t) + 1_{(f(y),f(x)]}(t))dt \\ &= \frac{1}{2}\sum_{x,y} b(x,y)|f(x) - f(y)|.\end{aligned}$$

Here, we use in the last step that the value of the integral is just $|f(x)-f(y)|$ (as can directly be seen by distinguishing the three cases $f(x) < f(y)$, $f(x) > f(y)$ und $f(x) = f(y)$).

The equality $T_2 = T_3$ holds. Obviously, $S_{K_t} = 0$ for $t \leq t_0$ (as the complement of K_t is the empty) and

$$K_t = \widetilde{K}_i \text{ for } t_{i-1} < t \leq t_i.$$

This easily gives the claim. □

Now, we consider the following situation. Let X be a finite set and $m : X \longrightarrow [0, \infty)$ be given. Let $K \subset X$ be given. Then, the *area* A_K of A is defined as

$$A_K := \sum_{x \in K} m(x).$$

Let furthermore (as above) for $f : X \longrightarrow \mathbb{R}$ and $t \in \mathbb{R}$ the set K_t be defined by

$$K_t := K_t^f := \{x \in X : f(x) \geq t\}.$$

THEOREM. *('Area Formel') Let $m : X \longrightarrow \mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ be given. Then,*

$$\sum_{x \in X} m(x)(f(x) - \min f) = \int_{\min f}^{\max f} A_{K_t} dt = \sum_{i=1}^{k}(t_i - t_{i-1})A_{K_{t_i}}.$$

Remarks.

- As in the case of the previous formula, it is easy to give an interpretation in terms of rectangles drawn in the plane. In this case, one draws intervals of length $m(x)$ for any $x \in X$ on the x-axis and considers the corresponding rectangle whose other side ranges from $\min f$ to $f(x)$.
- The upper boundary in the integration can be chosen as ∞ (as $A_{K_t} = 0$ for $t \geq \max f$).
- The lower boundary in the integration is important as $A_{K_t} = m(X) := \sum_{x\in X} m(x)$ for $t \leq \min f$. It is possible, however, to replace $\min f$ by any $c \leq \min f$. Then, one has to add $(\min f - c)m(x)$ in the last term.

Proof. We call the terms in the desired equality T_1, T_2, T_3 (in the order of appearance).

The equality $T_1 = T_2$ holds. We obviously have

$$A_{K_t} = \sum_{x\in K_t} m(x) = \sum_{x\in X} m(x) 1_{(-\infty, f(x)]}(t).$$

This gives

$$\int_{\min f}^{\infty} A_{K_t} dt = \sum_x m(x) \int_{\min f}^{\infty} 1_{(-\infty,f(x)]}(t) dt = \sum m(x)(f(x) - \min f)$$

and the desired claim follows.

The equality $T_2 = T_3$ holds. As in the previous proof we have

$$K_t = \widetilde{K}_i \text{ for } t_{i-1} < t \leq t_i$$

and the claim follows. □

Remarks.

- The sign of b does not play a role in the above considerations.
- Instead of K_t one could also consider $K_t^{'} := \{x : f(x) > t\}$.

2. Trading c for Dirichlet boundary conditions

In this section, we consider a general method to interchange x with $c(x) \neq 0$ by a Dirichlet boundary condition (i.e. a requirement of the form $f(x) = 0$). This will be useful later to reduce the situation of general graphs to graphs with $c \equiv 0$. This is relevant as we know the Co-Area formula only in the case $c \equiv 0$.

Replace x with $c(x) \neq 0$ by Dirichlet boundary condition: Let (b, c) be a graph over X. We now introduce the additional point ∞ and consider

$$\widetilde{X} := X \cup \{\infty\}$$

with

$$\widetilde{b} : \widetilde{X} \times \widetilde{X} \longrightarrow [0, \infty),\ \ \widetilde{b}(x,y) := \begin{cases} b(x,y) & : \ x, y \in X \\ c(x) & : \ x \in X, y = \infty \\ c(y) & : \ x = \infty, y \in X \\ 0 & : \ \textit{else} \end{cases}$$

Let $\widetilde{Q}$ be the form associated to the graph $(\widetilde{b}, 0)$ over $\widetilde{X}$. Then, obviously

$$Q(f) = \widetilde{Q}(\widetilde{f})$$

holds, where the function $\widetilde{f}$ on $\widetilde{X}$ is defined by $\widetilde{f}(x) = f(x)$ for $x \in X$ and $\widetilde{f}(\infty) = 0$. In this way, we have replaced a graph with $c \neq 0$ by a graph with vanishing c.

Replace Dirichlet boundary condition by x with $c(x) \neq 0$: Let (b, c) be a graph over X. Consider $K \subset X$ with $X \setminus K \neq \emptyset$. Then, any f on X with $f = 0$ on $X \setminus K$ can be considered as function f_K on K. The form Q induces on

$$\mathcal{H}_K := \text{functions on } K$$

the form Q_K defined by

$$Q_K(f, g) := Q(i_K f, i_K g)$$

with the canonical inclusion

$$i_K : \mathcal{H}_K \longrightarrow \mathcal{H}(= \text{functions on } X).$$

It is not hard to see that Q_K is again a Dirichletform. In fact, with

$$b_K : K \times K \longrightarrow [0, \infty), \;\; b_K(x, y) = b(x, y)$$

and

$$c_K : K \longrightarrow [0, \infty), \;\; c_K(x) = c(x) + \sum_{y \in X \setminus K} b(x, y)$$

we obtain by a short calculation (exercise)

$$Q_K(f, g) = \sum_{x \in K} b_K(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum c_K(x) f(x) g(x).$$

Note that the contributions to c_K arise from the values of c and the weights of those edges connecting K and $X \setminus K$.

It is not hard to see that the processes considered above are inverse to each other.

3. Cheeger inequality for Dirichlet forms on finite sets

In this section we use the considerations of the previous two sections to obtain a lower bound on Dirichletforms on finite graphs. This can be applied to obtain both lower bounds on the first eigenvalue (if $c \neq 0$) and on the second value (if $c = 0$). This will be discussed in subsequent sections.

Let (b, c) be a graph over X. To $K \subset X$ we associate the *surface* and the *area* as follows:

$$\begin{aligned} S_K &:= \sum_{x \in K, y \notin K} b(x, y) + \sum_{x \in K} c(x) \\ &= \text{Surface between } K \text{ and } K^c \text{ (measured in terms of } b \text{ and } c) \end{aligned}$$

$$\begin{aligned} A_K &:= \sum_{x\in K}\left(c(x)+\sum_{y\in X} b(x,y)\right)\\ &= \text{Area of } K \text{ (measured in terms of } b \text{ and } c) \end{aligned}$$

A special case is

$$m_x := m_{\{x\}} = c(x) + \sum_y b(x,y).$$

With this notation we have $A_K = \sum_{x\in K} m(x)$.
Note that in both cases c does play a role and can be considered as coding edges to ∞.

Remarks.

- (Exercise) An interpretation of the above expressions can be given in terms of Q as follows.

  $$S_K = Q(1_K) = Q_K(1) = \sum_{x\in K} c_K(x,x).$$

  If $c = 0$ this can also be seen to satisfy

  $$S_K = Q(1_K) = -Q(1_K, 1_{X\setminus K}).$$
- (Exercise) If (b,c) is connected, then $S_K > 0$ for all K with $K \neq X$ and $K \neq \emptyset$.
- We have defined both S_K and A_K via the b and c. In principle, one could also define them by suitable counting of vertices. This can be carried out as well and yields similar formulae.

The following result goes back to Dodziuk '84 and Dodziuk/Kendall '85.

THEOREM. *(General Cheeger inequality) Let (b,c) be a connected graph over X. Let $U \subset X$ with $U \neq \emptyset$ be given. Define*

$$\alpha := \min_{K\subset U, K\neq\emptyset} \frac{S_K}{A_K}.$$

Then,

$$Q(f) \geq \frac{\alpha^2}{2}\sum_{x\in X} m(x) f^2(x)$$

holds for all f vanishing outside of U.

Remark.

- For $U = X$ and $c = 0$, we have $\alpha = 0$ (chose $K = X$). Thus, in case of vanishing c the statement is only of interest for $U \neq X$. For such U the spectrum of the corresponding subgraph is 'lifted' due to introducting Dirichlet boundary conditions (see above).
- Note that the statement still makes sense on infinite graphs for f supported only on a finite set. In fact, it turns out that our proof works in this case as well.

Proof. Without loss of generality $f \geq 0$ (as $Q(f) \geq Q(|f|)$ and $\sum m(x) f(x)^2 = \sum m(x)|f(x)|^2$).
Without loss of generality we can assume that $c = 0$. (The reason is as follows: The case $U = X$ and $c = 0$ is trivial (as then $\alpha = 0$ holds). Thus, we assume that we are not in this case. We now replace the original graph by the network $(\widetilde{X}, \widetilde{b})$ with $\widetilde{X} = X, \widetilde{b} = b$ if $c = 0$ and $\widetilde{X}, \widetilde{b}$ given by the considerations of the previous section if $c \neq 0$. We also replace f by $\widetilde{f}$. The crucial observation is now that all quantities S_K and A_K keep their original values in the new graph by construction. It is just that absorbtion terms $c(x) \neq 0$ may now have become edges. Similarly, $\widetilde{Q}(\widetilde{f}) = Q(f)$ holds.)
From now on we will only consider the quantities with the tilde. Note that there exists an x with $\widetilde{f}(x) = 0$ (viz $x = \infty$ if $c \neq 0$ and $x \in X \setminus U$ if $c = 0$). In particular, $\min f = 0$.

The main trick is now to consider the term

$$A := \frac{1}{2} \sum_{x,y \in \widetilde{X}} \widetilde{b}(x,y) |\widetilde{f}^2(x) - \widetilde{f}^2(y)|.$$

We prove the upper bound

$$(UB) \quad A \leq Q(f)^{1/2} \sqrt{2} (\sum_{x \in X} m(x) f(x)^2)^{1/2}$$

and the lower bound

$$(LB) \quad A \geq \alpha \sum_{x \in X} m(x) f^2(x).$$

Taken together they give the claim.

Upper bound. We have $|\widetilde{f}^2(x) - \widetilde{f}^2(y)| = |\widetilde{f}(x) - \widetilde{f}(y)||\widetilde{f}(x) + \widetilde{f}(y)|$. The definition of A then gives

$$A = \frac{1}{2} \sum_{x,y} \widetilde{b}(x,y)^{1/2} |\widetilde{f}(x) - \widetilde{f}(y)| \widetilde{b}(x,y)^{1/2} |\widetilde{f}(x) + \widetilde{f}(y)|.$$

Applying Cauchy-Schwarz inequality and using

$$|\widetilde{f}(x) + \widetilde{f}(y)|^2 \leq 2\widetilde{f}(x)^2 + 2\widetilde{f}(y)^2$$

we obtain the claim. (Due to $\widetilde{f}(\infty) = 0$, the point ∞ does not contribute.)
Lower bound. Let $t_0 < t_1 < \ldots < t_M$ be the different values of f^2 and $\widetilde{K}_i := \{x : f(x)^2 \geq t_i\}$. By Co-area-formula applied to f^2 we have

$$A = \sum_{i=1}^{M} (t_i^2 - t_{i-1}^2) S_{\widetilde{K}_i}.$$

By definition of α we can estimate this as follows:

$$A \geq \alpha \sum_{i=1}^{M} (t_i^2 - t_{i-1}^2) A_{\widetilde{K}_i}.$$

Using $\min f = 0$ we obtain from the area-formual for the right hand side

$$\sum_{i=1}^{M} (t_i^2 - t_{i-1}^2) A_{\widetilde{K}_i} = \sum m(x) f^2(x).$$

This finishes the proof. □

Remark. The function $\sum_x m(x)f(x)^2$ is not the square of the norm of f. There are various ways to deal with this issue:

- Defining $d := \min\{m(x) : x \in X\}$ we obtain $\sum_x m(x)f(x)^2 \geq d\|f\|^2$.
- Instead of $\mathcal{H}$ one can consider the Hilbert space $\ell^2(X,m)$ of functions on X with inner product
$$\langle f,g\rangle_m := \sum_x f(x)g(x)m(x).$$
Then, $\sum_x m(x)f(x)^2 = \langle f,f\rangle_m = \|f\|_m^2$. The corresponding Laplacian is known as normalized Laplacian (exercise).
- It is possible to introduce a different Cheeger constant based on distance measurement with respect to intrinsic metrics. This has been done by Bauer / Keller / Wojciechowski '12.

4. Lower bounds on first and second eigenvalue

As an application of the result of the previous section we obtain a lower bound on the first eigenvalue (in case $c \neq 0$) and a lower bound on the second eigenvalue for general c. This lower bound on the second eigenvalue is particularly interesting if $c = 0$. In this case, it gives a lower bound on the spectral gap.

We start by discussing the bound on the first eigenvalue.

Corollary. *Let (b,c) be a connected graph over X with $c \neq 0$. Define $C := \sum_x c(x)$ and*
$$\beta := \min\{b(x,y) : b(x,y) \neq 0\} \cup \{C\}, \quad M := \sum_{x,y} b(x,y) + \sum_x c(x).$$
Then, $\alpha := \min_{K\subset X} \frac{S_K}{A_K}$ satisfies
$$\alpha \geq \frac{\beta}{M} > 0$$
and the smallest eigenvalue λ_1 of the operator L associated to the graph satisfies
$$\lambda_1 \geq \frac{\alpha^2}{2} d > 0$$
with
$$d := \min\{m(x) : x \in X\}.$$

Proof. It suffices to show the estimate on α. (Then, the remaining part of the claim follows from the previous theorem.)

Obviously, $A_K \leq M$ for all $K \subset X$. We now investigate S_K and consider two cases:

Case 1: $K = X$: By $c \neq 0$ we have
$$S_K = \sum_{x\in K, y\notin K} b(x,y) + \sum_{x\in K} (x) = \sum_{x\in X} c(x) \geq \beta.$$

Case 2: $K \neq X$: Then, $X \setminus K$ is not empty. As the graph is connected, there exists an $x \in K$ and $y \notin K$ with $b(x,y) > 0$. This gives $S_K \geq b(x,y) \geq \beta$. □

Remark.

- For $c = \kappa \delta_p$ for a $p \in X$ and $\kappa > 0$ we obtain
$$\lambda_1 \geq \frac{d}{2M^2} \kappa^2$$
for small values of κ. The infimum of the spectrum grows (at least) quadratic in the coupling. Of course, if $c \geq \kappa 1$, then $\lambda_1 \geq \kappa$ holds.
- If β and c only takes values in $\{0,1\}$, we obtain
$$\lambda_1 \geq \frac{1}{2M^2}$$
with $M :=$ 2 number of edges + number of x with $c(x) \neq 0$.

We now present a lower bound on the second eigenvalue. A main idea is that the eigenfunction to the second eigenvalue decomposes the set X into two part: the part where the eigenfunction is positive and the part where it is negative. On each of these parts we can apply Cheeger type estimates on the form.

The main step of the decomposition is done in the next proposition.

Proposition. *Let (b,c) be a connected graph over X with associated form Q and associated operator L. Let $\lambda_1 < \lambda_2 < ...\lambda_k$ be the different eigenvalues of L and φ an eigenfunction to λ_2. Let φ_+ and φ_- the positive and negative part of φ. Then, neither φ_+ nor φ_- vanish and*
$$Q(\varphi_+,\varphi_+) \leq \lambda_2 \|\varphi_+\|^2 \;\; und \;\; Q(\varphi_-,\varphi_-) \leq \lambda_2 \|\varphi_-\|^2.$$

Remark. The proposition effectively reduces an estimate of the second eigenvalue to an estimate of the first eigenvalue of a restricted form.

Proof. As the graph is connected, the eigenspace to the first eigenvalue is one-dimensional and spanned by a strictly positive eigenfunction (see above). Thus, neither φ_+ nor φ_- can vanish (as φ is orthogonal an the eigenfunction to λ_1). We prove the estimate on φ_+. The other estimate can be shown similarly (e.g. after multiplication with -1).
As Q is a Dirichlet form we have
$$Q(\varphi_+,\varphi_-) \leq 0$$
(see above). This gives
$$\begin{aligned} Q(\varphi_+,\varphi_+) &\leq Q(\varphi_+,\varphi_+) - Q(\varphi_+,\varphi_-) \\ &= Q(\varphi_+,\varphi) \\ &= \langle \varphi_+, L\varphi \rangle \\ &= \lambda_2 \langle \varphi_+, \varphi \rangle \\ &= \lambda_2 \|\varphi_+\|^2. \end{aligned}$$
This finishes the proof. □

We can now combine the proposition with the Cheeger inequality. This gives the following estimate on the second eigenvalue.

THEOREM. *Let (b, c) be a connected graph over X and let λ_2 be the second smallest eigenvalue of the associated operator L. Set*

$$\alpha := \min_{K \subset X, 0 < A_K \leq A_X/2} \frac{S_K}{A_K}$$

and $d := \min\{m(x) : x \in X\}$. Then, the estimate

$$\lambda_2 \geq \frac{\alpha^2}{2} d$$

holds.

Remark. In case $c = 0$ the value λ_2 is just the spectral gap $\lambda_2 = \lambda_2 - \lambda_1$. The preceding theorem then gives an estimate for the spectral gap (and hence the speed of convergence towards equilibrium in the heat equation). This is a main application of the theorem.

Proof. By the previous proposition, it suffices to give an estimate for $Q(\psi)$ for ψ supported on a set whose 'area' does not exceed $A_X/2$. Now, such an estimate immediately follows from the Cheeger inequality. This proves the theorem. □

5. Further remarks

We mention the following characterisation of the Cheeger constant. The proof can be given with the ideas developed above.

THEOREM. *(Variational characterisation of α) Let (X, b) be a network with $\sharp X \geq 2$. Let $m : X \longrightarrow (0, \infty)$ be given and define $A_K := \sum_{x \in K} m(x)$ for $K \subset X$ and*

$$\alpha = \alpha_m := \min\{\frac{S_K}{A_K} : 0 < A_K \leq A_X/2\}.$$

Then,

$$\alpha = \inf_{\varphi} \sup_{\gamma \in \mathbb{R}} \frac{\frac{1}{2}\sum_{x,y} b(x,y)|\varphi(x) - \varphi(y)|}{\sum_{x \in X} m(x)|\varphi(x) - \gamma|},$$

where the infimum is taken over all non-constant φ.

Remark.

- This is not a result in Hilbertspace but a minimization on ℓ^1. The corresponding minimization in Hilbertspace is related to the eigenvalues.
- One can replace φ by $\varphi - \textit{constant}$. This would not change anything.
- If for a given φ the number γ_{min} minimizes the denominator $\sum_{x \in X} |\varphi(x) - \gamma|$, then $\psi := \varphi - \gamma_{min}$ does not vanish and satisfies

$$\sum_{x \in X} |\psi(x) - \gamma| m(x) \geq \sum_{x \in X} |\psi(x)| m(x)$$

for all $\gamma \in \mathbb{R}$. Thus, it is possible to remove the supremum over the γ by considering the infimum over all non-vanishing φ with

$$\sum_{x\in X} |\varphi(x) - \gamma| m(x) \geq \sum_{x\in X} |\varphi(x)| m(x)$$

for all $\gamma \in \mathbb{R}$.

- The analogue in the continuum deals with

$$S = \inf \sup_{\gamma} \frac{\int_X |\nabla f| dx}{\int_X |f - \gamma| dx}.$$

This quantity is sometimes known as Sobolev constant. Inequalities of the form

$$\int_X |f - \gamma| dx \leq S \int_X |\nabla f| dx$$

with γ the average over f are called Poincaré Inequality.

We finally include a statement on an upper bound for the form.

Proposition. *Let (b, c) be a connected graph over X and define*

$$m : X \longrightarrow (0, \infty),\ m(x) = \sum_{y\in X} b(x, y) + c(x) \ \textit{and}\ D := \max m(x).$$

Then, we have

$$D \leq \|L\| \leq 2D$$

and, in particular,

$$Q(f) \leq 2D\|f\|^2.$$

Remark. The operator L is non-negative i.e selfadjoint with non-negative eigenvalues. Thus, $\|L\|$ is the largest eigenvalue of L. The proposition gives in particular that the spectrum of L is contained in $[0, 2D]$.

Proof. We prove the lower and the upper bound.

Lower bound: For $f = \delta_p$ with $p \in X$ we obtain

$$Q(f) = \frac{1}{2} \sum_{x,y} b(x, y)(\delta_p(x) - \delta_p(y))^2 + \sum_{x} \delta_p(x) c(x) = m(p).$$

As f is normalized Cauchy-Schwarz then yields

$$\|Lf\| = \|Lf\|\|f\| \geq Q(f, f) \geq m(p).$$

As this holds for all $p \in X$, we get $D \leq \|L\|$.

Upper bound: A direct calculation shows

$$
\begin{aligned}
Q(f) &= \frac{1}{2}\sum_{x,y} b(x,y)(f(x)-f(y))^2 + \sum_x c(x)f(x)^2 \\
&\leq \frac{1}{2}\sum_{x,y} b(x,y)(2f(x)^2+2f(y)^2) + \sum_x c(x)f(x)^2 \\
&= 2\sum_{x,y} b(x,y)f(x)^2 + \sum_x c(x)f(x)^2 \\
&\leq 2\sum_x f(x)^2(\sum_y b(x,y)+c(x)) \\
&= 2D\|f\|^2.
\end{aligned}
$$

This shows that the largest eigenvalue of L is bouded by $2D$ and the claim follows. □