Analysis on graphs

Course given at TU Graz March 2013 **Prof. Dr. D. Lenz**

Exercise Sheet I **Due to a later point of time**

(1) Let N be a positive integer. Equip $\mathbb{R}^N$ with Euclidean norm $\|\cdot\|$ and the space of real $N \times N$matrices with the norm

$$\|A\| := \max\{\|Af\| : \|f\| \leq 1\}.$$

Below convergence will always be understood with respect to these norms.

(a) Show that in this way indeed a norm is defined on the space of real $N \times N$ matrices. Show that this norm is submultiplicative i.e. that $\|AB\| \leq \|A\|\|B\|$ holds for any real $N \times N$ matrices A, B.

(b) Let A be a real $N \times N$ matrix. Show that the series $e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ converges absolutely.

(c) Let A be a real $N \times N$ matrix and define $P_t := e^{tA}$ for any $t \in \mathbb{R}$. Show that P is differentiable (i.e. that

$$\frac{d}{dt} P_t = \lim_{t \to t_0} \frac{1}{t - t_0} (P_t - P_{t_0})$$

exists for any $t_0 \in \mathbb{R}$) and that

$$\frac{d}{dt} P_t = AP_t = P_t A$$

holds.

(d) Let A and B be real $N \times N$ matrices with $AB = BA$. Show that $e^{A+B} = e^A e^B = e^B e^A$ holds.

(e) Let A and B be real $N \times N$ matrices. Prove the Trotter-Lie-product formula

$$e^{A+B} = \lim_{n \to \infty} \left(e^{\frac{1}{n}A} e^{\frac{1}{n}B} \right)^n.$$

(Hint: Set $S_n := e^{\frac{1}{n}(A+B)}$, $T_n := e^{\frac{1}{n}A} e^{\frac{1}{n}B}$.

* Show $\|S_n - T_n\| \leq C\frac{1}{n^2}$ (with a suitable constant C) by explicit consideration of the involved power series.
* Show $S_n^n - T_n^n = \sum_{m=0}^{n-1} S_n^m (S_n - T_n) T_n^{n-1-m}$ and conclude $\|S_n^n - T_n^n\| \leq C' \, n \, \|S_n - T_n\|$ with a suitable constant C'.

Use the preceeding steps to derive the desired formula.)

(f) What has to be changed if complex valued $N \times N$ matrices are considered?

(2) Let L be a real $N \times N$ matrix such that the semigroup $P_t := e^{-tL}$, $t \geq 0$, is positivity preserving. Show that the following two statments are equivalent:

(i) The semigroup P_t, $t \geq 0$, is positivity improving.

(ii) Only the trivial subspaces of $\mathbb{R}^N$ = Functions on $\{1, \ldots, N\}$ are invariant unter both the semigroup and multiplication by functions.

(3) Let (b, c) be a graph over $X = \{1, \ldots, N\}$ with $c \equiv 0$. Let L be the associated operator, λ_1 its smallest eigenvalue and Q the associated form. Show the following:

- $Q(f) \geq 0$ for all f and all eigenvalues of L are non-negative.
- $\lambda_1 = 0$.
- The eigenspace corresponding to λ_1 consists exactly of the functions which are constant on each connected component. In particular, if the graph is connected , then the eigenspace corresponding to λ_1 consists exactly of the constant functions.

(Hint: Show that for an eigenfunction f to the eigenvalue $\lambda_1 = 0$ one must have $0 = Q(f) = \sum_{x,y} b(x, y)(f(x) - f(y))^2$.)