Lessons from coin tosses

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In these notes we collect some rather surprising facts about consecutive coin tosses and discuss some implications for the real world. The presentation strives to use only elementary mathematical methods so that the proofs are accessible with little knowledge of probability theory.

First lessons

1. Waiting for a tie

Suppose that Zoe and Wael consecutively toss a fair coin. Each time there is heads, Wael pays 1 Euro to Zoe. If the coin shows tails, then Zoe pays 1 Euro to Wael. Note that the total gain of Zoe after round n is equal to the number of heads minus the number of tails in the first n tosses. We say that Zoe (Wael) is in the lead if her (his) gain is positive, and that there is a tie if the gain of both is equal to zero.

Question 1. Suppose that the first coin toss is heads, so that Zoe is in the lead after the first round. Zoe and Wael decide to toss the coin until there is again a tie between them. How many more rounds do they have to play *in average*?

In order to answer the question, let x be the expected number of rounds after the first toss until there is a tie again. If the second toss shows tails, then there is a tie already after round 2. In this case Zoe and Wael only need to play one further round until there is a tie.

Now suppose that the second toss is again heads. In that case Zoe leads by two points after the first two rounds. Before there is a tie, there must be again a time with a score, where Zoe leads by one point only. We say that the game's score is n if Zoe's gain is n. Note that the expected number of rounds needed until the game gets from score 2 back to score 1 is equal to x. Back at score 1 it takes in average x to attain score 0. Hence, conditional to the second toss to be heads, the expected number of rounds until the next tie is $1 + 2x$.

Since the second round shows heads and tails with probability 1/2, respectively, x satisfies the equation

$$x = \frac{1}{2}1 + \frac{1}{2}(1 + 2x). \tag{1}$$

Notice that Equation (1) implies $x = x+1$, which can only be true if $x = \infty$ or $x = -\infty$. Since x is non-negative, we can exclude the second possibility. This means, in average, Zoe and Wael have to play an infinite number of rounds until there is again a tie.

Let M_n denote Zoe's gain after round n. The process $M_1, M_2, M_3, \dots$ is a sequence of random variables that mathematicians refer to as a simple random walk. The random walk can be used to describe, in a stylized form, the evolution of many real processes. Prominent examples are economic factors, prices, population dynamics and random movements of particles in liquids or gases.

Let's discuss the implication of $x = \infty$ in an economic application. Suppose that a stock share price, say the price of a Siemens stock, evolves according to a random walk. Suppose that Wael buys on January 1st one hundred shares of Siemens. Assume further that by February the price per share has declined by one Euro, so that Wael has lost 100 Euro. Wael decides to keep the stock shares until the price per share has increased back to 100. If the Siemens price is a simple random walk, then in average Wael has to wait an infinite amount of time until his losses are made up again.

Wael's situation looks different if the coin is not fair, i.e. if the probability for showing heads differs from the probability of showing tails. Suppose that the probability p for showing heads satisfies $p < \frac{1}{2}$. In this case Wael is more likely to win a round than Zoe. The expected number of rounds x until there is again a tie satisfies

$$x = (1 - p) + p(1 + 2x). \tag{2}$$

Since $p < \frac{1}{2}$, we obtain $x = \frac{1}{1-2p}$. Notice that x is increasing in p with $\lim_{p\uparrow\frac{1}{2}} x = \infty$ and $\lim_{p\downarrow 0} x = 1$.

The lesson from this chapter is that you should not rely on randomness alone to make up for losses. Stick to losing investments (or projects) only if you have reasons to believe that there is a positive trend that will push your gain process back to a tie.

2. The St. Petersburg paradox

Somebody is offering you to play the following game: First you pay x Euro. Then a fair coin will be tossed. If there is heads, then you get 2 Euro. If there is tails, then you are allowed to toss again. If the second toss is heads, then you get 4 Euro, else you are allowed to toss the coin once more. If the third toss shows heads, then you get 8 Euros. You can toss the coin until it shows heads for the first time. If the first heads shows up at toss n, then you receive 2^n Euros.

Question 2. What is the highest price x that you are willing to pay for playing the game?

Let's denote the game's payoff by Y. Y is an example of a so-called random variable, i.e. a function with the set of possible scenarios as domain.

Note that Y is equal to 2 with prob $\frac{1}{2}$, equal to 4 with probability $\frac{1}{4}$, equal to 8 with probability $\frac{1}{8}$, and so on. Observe that the probability weights $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ sum up to 1. In particular, the probability that there is never heads is zero. So we can assume that the payoff Y takes only the finite values $2, 4, 8, \ldots$. What is the expected payoff? The expectation of Y is calculated by multiplying each possible outcome with the respective probability and then by adding up all products. The expected payoff thus satisfies

$$\frac{1}{2} * 2 + \frac{1}{4} * 4 + \frac{1}{8} * 8 + \cdots = 1 + 1 + 1 + \cdots = \infty.$$

We see that the expected amount of Euros you receive is equal to infinity. Nevertheless you are certainly not willing to pay a lot for playing the game. Most people's answers lie between 5 and 10 Euros.

The example shows that in our economic decisions we do not only strive to maximize expected income. We also care about risk. Risk aversion entails that we aim at keeping the variation in our income low.

3. Will there be a tie again?

The random variable describing the reward in the St. Petersburg paradox takes only finite values (the powers of 2). Its expectation, however, is infinite. This shows that a random variable can have infinite expectation although it takes only finite values.

Consider again Zoe and Wael playing the game of heads and tails of Section 1. Suppose that Zoe is in the lead after the first round and that both decide to toss the coin until there is again a tie. We have seen that the expected time until there is again a tie is infinite. An infinite expectation does not necessarily mean that with a positive probability the waiting time until the next tie is equal to infinity. Thus the following question arises:

Question 3. Can it happen, with a positive probability, that Zoe and Wael have to play infinitely long?

To answer the question, we use an argument similar to the one we have used for answering Question 1. Let p denote the probability that there will be a tie again. If the second toss shows tails, then there is a tie already after round 2. If the second toss is heads, then Zoe leads by two points after the first two rounds. Before there is a tie, there must be again a time with a score, where Zoe leads by one point only. The probability to get from score 2 back to score 1 at some point is equal to p. Back at score 1 the probability to attain score 0 at some point is p. Therefore, conditional to the second toss to be heads, the probability that there will be a tie again at some point is p^2.

Since the second round shows heads and tails with probability $1/2$, respectively, p satisfies the equation

$$p = \frac{1}{2} + \frac{1}{2}p^2. \tag{3}$$

Notice that p satisfies Equation (3) if and only if p satisfies $(p-1)^2 = 0$. Therefore, $p = 1$. In other words, there will be again a tie with probability one. The answer to question 3 is consequently no.

The lesson of this section is that there are events happening with probability 1, but it may take, in expectation, an infinite amount of time until they happen. We obtain thus a refinement of the message of Section 1: even if randomness will make up losses with probability one, it may not be reasonable to rely on it. When deciding to proceed in a project, do impose time constraints, not only probability constraints.

4. Long sequences of heads

Zoe and Wael decide to toss a coin 50 times, each time 1 Euro being at stake. At some point it happens that 5 heads show up in a row. Wael complains that he has a streak of bad luck.

Question 4. What is the probability that in a sequence of 50 coin tosses there is a row of at least 5 subsequent heads?

We take for a while a more general perspective and explain how one can calculate the probability that in sequence of n coin tosses one can find at most k consecutive heads. In the following we call a subsequence of consecutive heads a run.

Let $A_n(x)$ be the number of sequences of length n having no run of heads longer than x. There is no general formula for $A_n(x)$. However, one can calculate $A_n(x)$ recursively.

Any sequence in $A_n(x)$ starts either with T or with a run of heads with length $\leq x$. Note that there are exactly $A_{n-1}(x)$ sequences in $A_n(x)$ starting with T, $A_{n-2}(x)$ sequences starting with HT, $A_{n-3}(x)$ sequences starting with HHT, and so on. We therefore have

$$A_n(x) = A_{n-1}(x) + A_{n-2}(x) + ... + A_{n-x-1}(x) = \sum_{j=0}^{x} A_{n-1-j}(x). \tag{4}$$

The probability that in a sequence of length n there are no more than x heads in a row is given by

$$w(n,x) = \frac{A_n(x)}{2^n}. \tag{5}$$

We refer to Appendix C for a Python program calculating $w(n,x)$.

Let's come back to our specific question and use (5) to determine the probability that in sequence of 50 tosses there is a run of 5 consecutive heads. Using the program of Appendix C we obtain that $w(50,4) \approx 0,45$. This means that it happens with a probability of $0,55$ that at some point during the 50 tosses there is a run of at least 5 heads in a row. It is more likely that there will be a run of 5 heads than not! Hence Wael's complaint has no foundation.

This section shows that in coin tosses the probability for having a streak of bad luck is surprisingly large. Suppose the success of your efforts is determined via coin tosses. The lesson is that it is very likely that in some periods the odds are against you and that you don't make progress. Sometimes, however, chance is in your favor and you are crowned with success. Most importantly, don't despair in a streak of bad luck.

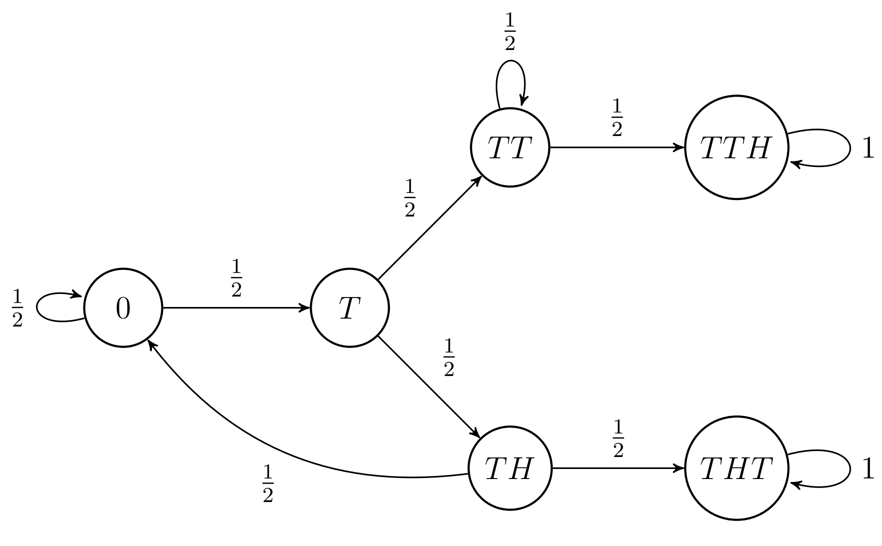
To give an explicit example, consider the performance of a sports team over a whole championship season. The probability of losing 5 games in a row just by bad luck, but not because of a drop in the performance potential, is substantial. Trainers should not be fired just because their team loses several games in a row.

5. What comes first: TTH or THT?

Zoe and Wael play the following game: they consecutively toss a coin until for the first time either the sequence "TTH" or "THT" appears. In the first case Zoe wins, in the second Wael. For example, if the first five coin tosses are "HHTHT", then Wael wins.

Question 5. With which probability does Zoe win the game? In other words, with which probability does the sequence "TTH" appear before the sequence "THT"?

The game has essentially the following 6 possible states: "0", "T", TT", "TH", "TTH", "THT". The state "0" corresponds to the start. If the first coin toss is tails, then the first element of both sequences "TTH" and "THT" has appeared and we are in the state "T". If the first coin toss is heads, then we are at the start of the game again. The state "T" is followed either by "TT" or "TH", each with probability 1/2. From "TT" we arrive with probability 1/2 in "TTH"; with probability 1/2 we have another tails, which means that the game is still in the state "TT". We illustrate all possible transitions and the corresponding probabilities in the following graph.



The dynamics of the game is an example of a so-called finite state *Markov chain*. The Markov chain here has 6 states, where the two states "TTH" and "THT" are absorbing.

Let h be the probability that Zoe wins the game. Then h is equal to the probability that the Markov chain ends up in the absorbing state "TTH". There are standard methods for determining the absorbing probability (see e.g. Chapter 4 in [6] or Satz 6.9 in [4]). In the following, we use a direct method for determining h.

The probability for attaining the state "TT" before the state "TH" is $\frac{1}{2}$. Note that it is not possible to get from "TT" to "THT". Therefore, once we are in "TT", we are sure to be absorbed by "TTH" sooner or later. In particular, with probability $\frac{1}{2}$ the Markov chain attains "TTH" without ever passing through the state "TH".

Now suppose that we attain "TH" before "TT". If the subsequent coin toss is tails, then we end up in the absorbing state "THT". If the subsequent coin toss is heads, then we arrive back at the start. At the start the probability of ending up in the absorbing state "TTH" is again h. Notice that the probability of coming back to the start is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Therefore, h satisfies the equation

$$h = \frac{1}{2} + \frac{1}{4}h.$$

This implies $h = \frac{2}{3}$. We have thus shown that Zoe wins with probability $\frac{2}{3}$.

For two finite sequences A and B we write $A \prec B$ if B appears before A with a probability strictly larger than $\frac{1}{2}$. It is remarkable that $\prec$ is not a transitive relation on the set of 3-term sequences (transitive means that if $A \prec B$ and $B \prec C$, then $A \prec C$). Indeed, one can show that

$$TTH \prec HTT \prec HHT \prec THH \prec TTH, \tag{6}$$

and hence $\prec$ can not be transitive.

Another surprising fact is that for any 3-term sequence A we can find another 3-term sequence B with $A \prec B$ (see Chapter 5 on "Non-transitive paradoxes" in [3]). Therefore, it is not a fair game if one player can choose a 3-term sequence after the other player. This game is sometimes referred to as Penney's game.

I am not aware of real-world implications of the example presented in this section - except that it nicely illustrates that probabilities can be counterintuitive.

6. Who goes broke first?

Zoe and Wael play again the simple version of the coin tossing game: each time there is heads, Wael pays 1 Euro to Zoe. If the coin shows tails, then Zoe pays 1 Euro to Wael. Suppose that Zoe starts the game with a Euros in her pocket and Wael with b Euros. They play until one of them goes bankrupt, that is to say until the total fortune of either Zoe or Wael attains the level 0.

Question 6. What is the probability that Zoe wins? Said otherwise, what is the probability that, starting with a Euros, Zoe's fortune attains the level $a + b$ before reaching 0? Let us write this probability $P(a, a + b)$.

Notice that by definition, $P(0,b)=0$ and $P(a,a)=1$. Assume that the probability of showing heads is p and the probability of showing tails is $q=1-p$.

If the first toss is heads, then Zoe's fortune becomes $a+1$ and the probability that she wins the game with $a+1$ Euros in her pocket is given by $P(a+1,a+b)$. If the first toss is tail, then Zoe's fortune becomes $a-1$ and her probability of winning is $P(a-1,a+b)$. Since the first round shows heads and tails with respective probabilities p and q, $P(a,a+b)$ satisfies the equation

$$P(a,a+b)=pP(a+1,a+b)+qP(a-1,a+b).$$

Let us write $P(x)$ instead of $P(x,a+b)$ for $x\in\{0,\ldots,a+b\}$ to simplify the notation. If we replace $P(a)$ in the previous equation by $pP(a)+qP(a)$, we get

$$P(a+1)-P(a)=\frac{q}{p}\left(P(a)-P(a-1)\right).$$

Hence we have proved that $(P(a+1)-P(a))_{a\in\mathbb{N}}$ is a geometric sequence with common ratio $\frac{q}{p}$:

$$P(a+1)-P(a)=\left(\frac{q}{p}\right)^a(P(1)-P(0))=\left(\frac{q}{p}\right)^a P(1),$$

since $P(0)=P(0,b)=0$. To compute the value $P(1)$, first notice that

$$\sum_{j=1}^{a-1}(P(j+1)-P(j))=P(a)-P(1),$$

which yields

$$\begin{aligned}P(a)&=P(1)+P(1)\sum_{j=1}^{a-1}\left(\frac{q}{p}\right)^j\\&=\begin{cases}P(1)\frac{1-(\frac{q}{p})^a}{1-\frac{q}{p}} & \text{if } p\neq q\\ P(1)a & \text{if } p=q=\frac{1}{2}.\end{cases}\end{aligned}$$

Recall that $P(a)$ is a notation for $P(a,a+b)$. The right hand side of the previous equation only depends on b through the term $P(1,a+b)$. Taking $b=0$, and using the fact that $P(a,a)=1$, we obtain

$$1=P(a,a)=\begin{cases}P(1,a)\frac{1-(\frac{q}{p})^a}{1-\frac{q}{p}} & \text{if } p\neq q\\ P(1,a)a & \text{if } p=q=\frac{1}{2}\end{cases}$$

and then, $\forall a\in\mathbb{N}^*$,

$$P(1,a)=\begin{cases}\frac{1-\frac{q}{p}}{1-(\frac{q}{p})^a} & \text{if } p\neq q\\ \frac{1}{a} & \text{if } p=q=\frac{1}{2}.\end{cases}$$

So we have proved that the probability that Zoe wins the game is given by

$$P(a, a+b) = \begin{cases} \frac{1-(\frac{q}{p})^a}{1-\left(\frac{q}{p}\right)^{a+b}} & \text{if } p \neq q \\ \frac{a}{a+b} & \text{if } p = q = \frac{1}{2}. \end{cases}$$

Now suppose that Zoe and Wael each have 20 Euros before starting the game: $a = 20$, $b = 20$. If $p = \frac{1}{2}$, then the probability that Zoe wins is $\frac{1}{2}$. Assume now that $p = 55\%$, so that at each coin tossing, Zoe has slightly more chances to win, in that case the probability that she wins the game is 98%! And it becomes $99,9699\%$ if $p = 60\%$. As shown in Figure 1, the probability that Zoe wins increases very fast when p takes values larger than $\frac{1}{2}$.

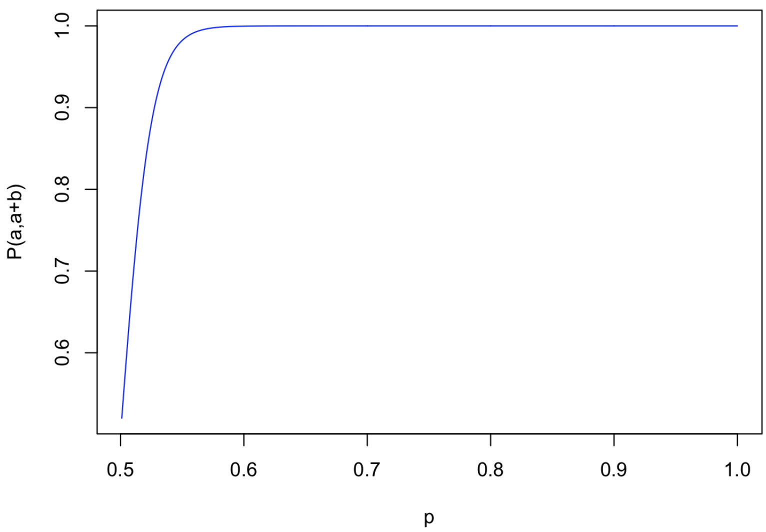


Figure 1: The function $p \mapsto P(20, 20+20)$, $p > \frac{1}{2}$.

Another surprising fact is the following: if b takes arbitrarily big values, and if $p > q$, then $1 - \left(\frac{q}{p}\right)^{a+b}$ becomes close to one. This means that the probability that Zoe wins is approximatively given by $1 - (\frac{q}{p})^a$, no matter how much money Wael brings with him! For instance, if Zoe starts the game with 20 Euros, Wael starts with 1 million Euros and $p = 55\%$, the probability that Wael goes broke before Zoe is $98,2\%$!!

This section shows that you should never go play a game in a Casino, where the chances to win are slightly less than $\frac{1}{2}$ (for example in the Roulette game), no matter how much money you bring with you.

Lessons with more advanced probability methods

7. A mathematical model for coin tosses

To formulate some of the next questions, and in particular to prove their answers, a precise mathematical model for a sequence of coin tosses is helpful. In this section, therefore, we do not answer a question, but describe such a model.

Let $\Omega_n = \{H, T\}^n$ be the set of all heads and tails sequences of length $n \in \mathbb{N}$. We denote the ith element of any $\omega \in \Omega_n$ by ω_i, where $i \in \{1, \ldots, n\}$. Moreover, for every i we define a mapping $X_i : \Omega_n \to \{-1, 1\}$ by setting

$$X_i(\omega) = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T. \end{cases}$$

Let $M_0 = 0$ and $M_k = X_1 + \ldots + X_k$, $k \in \{1, \ldots, n\}$. Observe that M_k is equal to Zoe's total gain up to time k. To illustrate this, let $n = 5$ and $\omega = HHTTH$. Then $M_1(\omega) = 1$, $M_2(\omega) = 2$, $M_3(\omega) = 1$, $M_4(\omega) = 0$ and $M_5(\omega) = 1$, as illustrated in Figure 2. The sequence $(M_k)_{k\in\{0,\ldots,n\}}$ is the precise mathematical definition of the simple random walk (cf. Section 1).

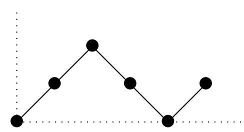


Figure 2: The map $k \mapsto M_k(\omega)$, with $\omega = HHTTH$

Let $\mathcal{P}_n$ be the power set (i.e. the collection of all subsets) of Ω_n. Let $P_n : \mathcal{P}_n \to [0, 1]$ be the mapping defined by

$$P_n(A) = \frac{1}{2^n}|A|,$$

where $|A|$ denotes the number of elements in A. We refer to the elements of $\mathcal{P}_n$ as events and call the map P_n a probability measure. Notice that for each sequence ω in Ω_n we have $P_n(\{\omega\}) = \frac{1}{2^n}$. Thus, P_n assigns to every head and tail sequence the same probability, and thus reflects that the coin is fair.

8. The arcsine law

Let's stay with Zoe and Wael playing the game of heads and tails of Section 1.

Question 7. Let p_n be the probability that Zoe's gain process spends at least 80% of the first n time units above zero. Where does p_n converge to as $n \to \infty$?

First we need to define exactly what we mean by saying that Zoe's gain process spends more than 80% of the time above zero.

In order to give a rigorous definition, we introduce the notion of a path. A *path* between 0 and $n \in \mathbb{N}$ is a mapping $f : \{0, 1, \ldots, n\} \to \mathbb{Z}$ such that $|f(k+1) - f(k)| = 1$, for all $k < n$. For $x, y \in \mathbb{Z}$ we denote by $\mathcal{Q}_n(0, x)$ the set of paths f between 0 and n with $f(0) = x$, and by $\mathcal{Q}((0, x), (n, y))$ the set of paths between 0 and n with $f(0) = x$ and $f(n) = y$.

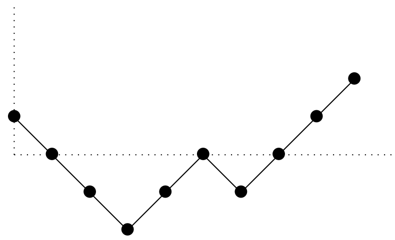


Figure 3: An example of path in $\mathcal{Q}((0,1),(9,2))$

To any $\omega \in \Omega_n$ we assign the path $k \mapsto M_k(\omega)$, $k \in \{0, \ldots, n\}$. Observe that the mapping is a one-to-one correspondence between Ω_n and $\mathcal{Q}_n(0, 0)$. Therefore, the number of elements in an event $A \subset \Omega_n$ is equal to the number of corresponding paths in $\mathcal{Q}_n(0, 0)$.

We next collect some combinatorial results on paths.

Lemma 8.1. *We have*

a) $|\mathcal{Q}_n(0, 0)| = 2^n$*;*

b) if m *is odd, then* $|\mathcal{Q}((0, 0), (2n, m)| = 0$*;*

c) for $-n \leq k \leq n$ *we have* $|\mathcal{Q}((0, 0), (2n, 2k)| = \binom{2n}{n-k}$.

Proof. Exercise. □

There is also an explicit formula for the number of non-negative paths, that means the paths that never go below zero.

Lemma 8.2. *There are* $\binom{2n}{n}$ *non-negative paths in* $\mathcal{Q}_{2n}(0, 0)$.

Proof. See Appendix A. □

Given a path f, we refer to the straight line between $(k, f(k))$ and $(k+1, f(k+1))$ as the segment of f between k and $k+1$. We say that it is non-negative if it lies above the x-axis. Note that the segment is non-negative if and only if $\min\{f(k), f(k+1)\} \geq 0$.

We define the time a path spends above zero as the number of non-negative path segments. For example, if the first ten tosses are "THHHTTHHTH", then Zoe's gain process spends 8 out of the first 10 time units above zero, as illustrated in Figure 4.

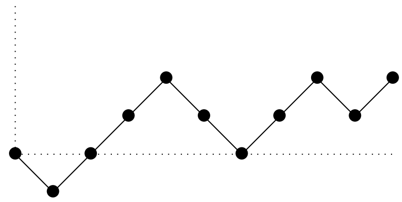


Figure 4: 8 out of the first 10 time units above zero

We first calculate the probability that Zoe's gain process spends at least 80% of the first ten time units above zero.

Notice that by Lemma 8.2 there are $\binom{10}{5}$ non-negative paths of length 10. Let d_k denote the number of paths of length 10 where the segments between k and $k+2$ are below and all other segments above the x-axis.

It is straightforward to see that $d_0 = \binom{8}{4}$, $d_2 = \binom{6}{3}$ and $d_4 = 2\binom{4}{2}$. In order to calculate d_6 note first that, by formula (23), the number of non-negative paths in $\mathcal{Q}((0,0),(6,0))$ is given by 5. Since there are two non-negative paths between 8 and 10 starting in zero, we have obtain $d_6 = 5 * 2$. Finally, the number of non-negative paths in $\mathcal{Q}((0,0),(8,0))$ is given by 14 (see again formula (23)), and hence $d_8 = 14 * 2$. To sum up, the number of paths in $\mathcal{Q}_{10}(0,0)$ with at least 8 non-negative segments is

$$\binom{10}{5} + \binom{8}{4} + \binom{6}{3} + 2\binom{4}{2} + 10 + 28 = 392.$$

Therefore, $p_{10} = 392/2^{10} \approx 0,38$.

Surprisingly, p_n does not converge to zero as $n \to \infty$. Indeed, we have $\lim_n p_n = 1 - \frac{2}{\pi}\arcsin(\sqrt{0.8}) \approx 0,3$.

More generally, we have the following result, known as the arcsine law.

Proposition 8.3. *Let $x \in [0,1]$ and denote by $A_n(x)$ the event that, up to time n, at least a fraction of x time units in Zoe's gain process is above zero. Then*

$$\lim_n P_n(A_n(x)) = \int_x^1 \frac{1}{\pi}\frac{1}{\sqrt{y(1-y)}}dy. \tag{7}$$

The integral can be shown to be equal to $1 - \frac{2}{\pi}\arcsin(\sqrt{x})$, explaining the name "arcsine law".

Before we prove the proposition, let's discuss what lesson we can draw from it. Figure 5 shows the graph of the function under the integral in (7). The proposition implies that, after long sequences of coin tosses, the distribution of the fraction of time spent above zero it is more likely to attain an extreme value close to 0 and 1 than a value close to $\frac{1}{2}$. In other words, it is more likely that Zoe or Wael is ahead most of the time than that both are ahead for approximately the same amount of time.

The proposition shows that in any competition with a random walk like evolution it is very likely that one competitor is ahead for most of the time. This is even the case when all competitors are equally strong (likely to score).

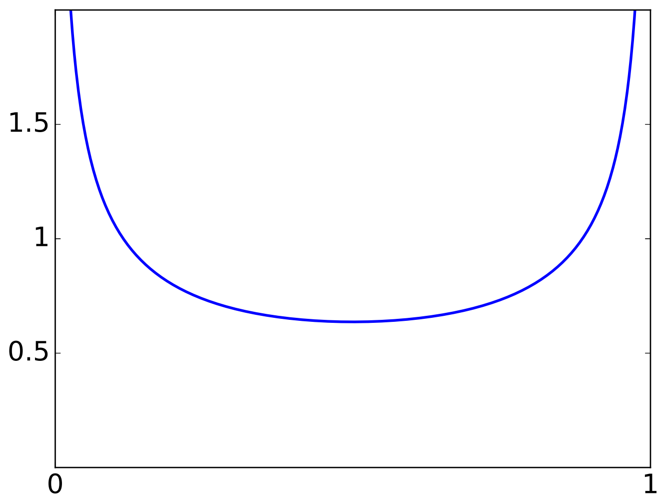


Figure 5: The density of the arcsine law.

Consider ten monkeys, each of them managing an asset fund by choosing randomly an investment strategy. The arcsine law indicates that it is very likely that one of ten funds will perform better for most of the time than the other funds. If you didn't know that a monkey manages the best performing fund, you could easily believe that it is managed by an expert and naively purchase some shares.

Now let's turn to the proof of Proposition 8.3. We first introduce some notation. Let $u_{2n} = \binom{2n}{n}$. Recall that u_{2n} is equal to $|\mathcal{Q}((0,0),(2n,0))|$ (Lemma 8.1) and also equal to the number of non-negative paths in $\mathcal{Q}_{2n}(0,0)$ (Lemma 8.2). Besides, let

$$w_{2n} = |\{f \in \mathcal{Q}((0,0),(2n,0)) : f(j) > 0 \text{ for all } j = 1,\dots,2n-1\}|.$$

Lemma 8.4. *We have*

$$w_{2n+2} = \frac{1}{n+1} u_{2n}, \tag{8}$$

and

$$w_{2n+2} = \sum_{k=1}^{n} w_{2k} w_{2n-2k+2}. \tag{9}$$

Proof. Let B be the set of non-negative paths in $\mathcal{Q}((0,0),(2n,0))$. Moreover, let $C = \{f \in \mathcal{Q}((0,0),(2n+2,0)) : f(j) > 0 \text{ for all } j = 1,\dots,2n+1\}$. Note that all f in C have the same first and last segment. Deleting the first and the last segment provides a one-to-one correspondance between C and B. With Equation (23) of Lemma 8.2, applied to $k=0$, we obtain $|C| = |B| = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} u_{2n}$, and hence (8).

In order to prove (9), let

$$B_r = \{f \in B : f(2r) = 0 \text{ and } f(k) > 0 \text{ for all } k < 2r\},$$

$r \in \{1, \ldots, n\}$. We split any path f in B_r at $2r$. Note that the part up to time $2r$ is always stricly above zero between 0 and $2r$. The part between $2r$ and $2n$, however, can take the value zero at any even time. There are w_{2r} paths with lenghts $2r$ satisfying $f(2r) = 0$ and $f(k) > 0$ for all $k \in \{1, \ldots 2r - 1\}$. Moreover, by the first part of the proof, there are $w_{2n-2r+2}$ non-negative paths g of length $2n-2r$ satisfying $g(2n-2r) = 0$. Therefore, $|B_r| = w_{2r}w_{2n-2r+2}$. Since B is the disjoint union of $B_1, \ldots, B_n$, we obtain (9). □

Note that $f \in \mathcal{Q}_{2n}(0,0)$ can take the value zero only at even time points. A combinatorial argument shows that

$$u_{2n} = \sum_{r=1}^{n} 2w_{2r}u_{2n-2r}, \tag{10}$$

where we interpret $2r$ as the first time where the path returns to the level 0. For $0 \leq k \leq n$ let $b_{2k,2n}$ be the number of paths in $\mathcal{Q}_{2n}(0,0)$ with exactly $2k$ segments above zero.

Lemma 8.5. *We have*

$$b_{2k,2n} = u_{2k}u_{2n-2k}. \tag{11}$$

Proof. Note that (11) holds true for $k = 0$, since $u_0 = 1$ and by Lemma 8.2, $b_{2n,2n} = \binom{2n}{n} = u_{2n}$.

Similarly, (11) holds true for $k = n$, since $b_{0,2n} = \binom{2n}{n} = u_{2n}$.

To prove the statement for $k \in \{1, \ldots, n-1\}$ we make an induction along n.

$\underline{n = 1}$: See first part of the proof.

$\underline{n - 1 \to n}$: Let $1 \leq k < n$. Every path in $\mathcal{Q}_{2n}(0,0)$ with exactly $2k$ non-negative segments returns to zero before $2n$. Any possible first return time is an even number. Suppose the first return time is $2r$. There are $2w_{2r}$ paths between 0 and $2r$ returning to zero for the first time at $2r$. Half of these paths spend all time units until $2r$ above zero, half of the paths stay always below. Summing over all possible return times, we obtain

$$b_{2k,2n} = \sum_{r=1}^{k} w_{2r}b_{2k-2r,2n-2r} + \sum_{r=1}^{n-k} w_{2r}b_{2k,2n-2r}.$$

By induction hypothesis and Equation (10) this implies

$$
\begin{aligned}
b_{2k,2n} =& \sum_{r=1}^{k} w_{2r} u_{2k-2r} u_{2n-2k} + \sum_{r=1}^{n-k} w_{2r} u_{2k} u_{2n-2r-2k} \\
=& u_{2n-2k} \sum_{r=1}^{k} w_{2r} u_{2k-2r} + u_{2k} \sum_{r=1}^{n-k} w_{2r} u_{2n-2r-2k} \\
=& \frac{1}{2} u_{2n-2k} u_{2k} + \frac{1}{2} u_{2k} u_{2n-2k} \\
=& u_{2k} u_{2n-2k}.
\end{aligned}
$$

□

We have now everything at hand for proving the arcsine law.

Proof of Proposition 8.3. Let T_n be the number of time units until n that Zoe's gain process (M_k) spends above zero. Equation (11) implies that

$$
P_{2n}(T_{2n} = 2k) = \frac{1}{2^{2n}} b_{2k,2n} = \frac{1}{2^{2n}} u_{2k} u_{2n-2k}, \tag{12}
$$

for $1 \leq k \leq n$. Next we use Stirling's formula (see Lemma B.1 below) [1]

$$
n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}. \tag{13}
$$

In particular, this means that $\frac{\sqrt{\pi}\sqrt{n} U_{2n}}{2^{2n}} \longrightarrow 1$, as $n \to \infty$. Therefore $\frac{U_{2n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi}\sqrt{n}}$.

Let $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, $x \in (0,1)$. Then $P(T_{2n} = 2k) = \frac{1}{2^{2n}} u_{2k} u_{2n-2k} \approx \frac{1}{n} f(k/n)$.

Finally, using properties of the Riemann integral, we get

$$
\begin{aligned}
\lim_n P_{2n}(A_{2n}(x)) &= \lim_n \sum_{k: k \leq n \text{ and } \frac{2k}{2n} \geq x} P_{2n}(T_{2n} = 2k) \\
&= \lim_n \sum_{k: k \leq n \text{ and } \frac{k}{n} \geq x} \frac{1}{n} f(k/n) \\
&= \int_x^1 f(y) dy = 1 - \frac{2}{\pi} \arcsin(\sqrt{x}).
\end{aligned}
$$

Since the contribution of a single coin toss to the fraction of time units spent above zero becomes arbitrarily small as $n \to \infty$, we have $\lim_n (P_{2n}(A_{2n}(x)) - P_{2n+1}(A_{2n+1}(x)) = 0$, and hence (7). □

[1] Recall that for two positive real sequences (a_n) and (b_n) we write $a_n \sim b_n$ if $\lim_n \frac{a_n}{b_n} = 1$.

9. Double stakes when the gain is negative

Suppose that Zoe and Wael play again the game of heads and tails. In addition to before, suppose that Zoe can choose in every round whether 1 or 2 Euros are at stake. If 2 Euros are at stake, then she gets 2 Euros or pays 2 Euros when the coin lands heads or tails, respectively.

Suppose that Zoe places a stake of 2 whenever she is behind and a stake of 1 whenever she is in the lead or there is a tie. In other words, she chooses the risky option when she is behind and the safe option if she is ahead. We can briefly describe her strategy with the words: "play safe if ahead and take risk if behind". In the following we refer to her strategy as the 1,2-strategy.

Let's model Zoe's gain process when she is playing the 1,2-strategy.

Let $Y_0 = 0$ and

$$Y_{k+1} = \begin{cases} Y_k + X_k, & \text{if } Y_k \geq 0, \\ Y_k + 2X_k, & \text{if } Y_k < 0. \end{cases}$$

Note that Y_k describes Zoe's gain after round k. We call the process $(Y_k)_{k\in\{0,\ldots,n\}}$ a 1,2-random walk.

Question 8. What is the probability for Zoe to be ahead in the long run, i.e. what is $\lim_{n\to\infty} P_n(Y_n \geq 0)$?

It is clear that the simple random walk (M_n), making steps only of size one, satisfies $\lim_n P(M_n \geq 0) = \frac{1}{2}$. Rather surprisingly, the limiting probability for the 1,2-random walk is different.

Proposition 9.1. *We have* $\lim_{n\to\infty} P_n(Y_n \geq 0) = \frac{2}{3}$.

What's the lesson? By playing safe when ahead and taking risk when behind one can indeed increase the probability of being on the winner side at the end of the game.

Suppose that in each round not Euros but points are at stake and that Zoe receives a prize, say 10 Euros, if she has collected a non-negative amount of points after n rounds. If always the same amount of points was at stake, then Zoe would win the prize approximately with probability $\frac{1}{2}$. By choosing the 1,2-strategy she increases her winning chances to $\frac{2}{3}$, and increase of 33%.

In professional sports games one often observes that teams that are behind take more risk and teams that are ahead play safe (in particular towards the end of the game). Proposition 9.1 indicates that this behavior is indeed optimal for maximizing the probability of winning.

As a further real-world example consider a manager of a firm who can choose between risky and safe business decisions. Suppose that the manager gets a bonus at the end of the year if the firm value is greater than or equal to a reference index. Suppose that the firm value minus the reference index is a process evolving like a random walk, where risky decisions imply steps of size two and safe decisions steps of size one. The manager has a strong incentive to choose the 1,2-strategy over the strategy to make always safe decisions. Indeed, with the 1,2-strategy the probability of getting the bonus is considerably larger.

The last example illustrates that bonus schemes can incentivize managers to play safe if ahead and take risk if behind. We remark that this is usually not in the interest of the firm's owner, since taking large risks on the loosing side can entail huge losses. Indeed, the expectation of the absolute value $|Y_n|$, conditional on the 1,2-random walk to be below zero, is larger than its expectation conditional to be above zero. This also explains that although Y_n takes negative values with a smaller probability than positive values, its expectation is zero.

Now let's turn to the proof of Proposition 9.1. We first derive an explicit formula for $P(Y_n \geq 0)$. We need to adapt the notion of a path. A *1,2-path* of length n is a function $f\{0, 1, \ldots, n\} \to \mathbb{Z}$ such that $f(0) = 0$ and

$$|f(k+1) - f(k)| = \begin{cases} 1, & \text{if } f(k) \geq 0, \\ 2, & \text{if } f(k) < 0, \end{cases}$$

for all $k \in \{0, \ldots, n-1\}$. For every scenario $\omega \in \Omega_n$, Zoe's gain process $k \mapsto Y_k(\omega)$, $k \in \{0, \ldots, n\}$ is a 1,2-path of length n. Observe that there is a one-to-one correspondance between Ω_n and the set of all 1,2-paths.

In the following we denote by a_n the number of 1,2-paths of length n that are non-negative at time n. Observe that a_n coincides with $|\{M_n \geq 0\}|$.

Proposition 9.2. *We have* $a_n = \frac{1}{3}(2^{n+1} + (-1)^n)$.

Note that Proposition 9.2 implies Proposition 9.1, since $\lim_{n\to\infty} P_n(M_n \geq 0) = \lim_{n\to\infty} \frac{a_n}{2^n} = \frac{2}{3}$.

In order to prove Proposition 9.2 we introduce some definitions:

- w_{2n} = the number of paths of length $2n$ satisfying $f(1) > 0, \ldots, f(2n-1) > 0, f(2n) = 0$
- u_k = the number of paths of length k that stay above the x-axis
- ζ_k = the number of positive paths of length k that never return to zero

Note that a non-negative 1,2-path is a non-negative path in the sense of Section 8. Therefore, by Lemma 8.2 we have $u_{2n} = \binom{2n}{n}$ and by Lemma 8.4 we have $w_{2n} = \frac{1}{n}\binom{2(n-1)}{n-1}$. The last number is also referred to as a Catalan number (see [8] for a whole book on Catalan numbers). More precisely, the Catalan numbers are defined as $c_0 = 1$ and $c_n := \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 1$. Thus we have $w_{2n} = c_{n-1}$. The Catalan numbers have the following nice properties.

Lemma 9.3. *For all* $n \geq 0$ *we have*

$$c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}, \tag{14}$$

and for all $n \geq 1$

$$\sum_{k=1}^{n-1} c_k 2^{2n-2k} = 2^{2n} - 2\binom{2n}{n}. \tag{15}$$

Proof. Equation (14) follows from Equation (9).

We prove Equation (15) via induction:

n = 1: both sides are equal to zero.

$n \to n+1$: Suppose (15) holds true for n. Then

$$\begin{aligned}\sum_{k=1}^{n} c_k 2^{2(n+1)-2k} &= 4c_n + 4\sum_{k=1}^{n-1} c_k 2^{2n-2k} \\ &\overset{I.H.}{=} \frac{4}{n+1}\binom{2n}{n} + 4\left(2^{2n} - 2\binom{2n}{n}\right) \\ &= 2^{2(n+1)} + 4\left(\frac{1}{n+1} - 2\right)\binom{2n}{n} \\ &= 2^{2(n+1)} - 2\binom{2(n+1)}{n+1}.\end{aligned}$$

□

Observe that

$$u_{2n+1} = 2u_{2n} - c_n = (2 - \frac{1}{n+1})\binom{2n}{n} = \frac{1}{2}\binom{2(n+1)}{n+1}.$$

Notice that $\zeta_k = u_{k-1}$, and hence

$$\zeta_{2n} = \frac{1}{2}\binom{2n}{n} \text{ and } \zeta_{2n+1} = \binom{2n}{n}.$$

A combinatorial argument shows

$$u_{2n} = \zeta_{2n} + \sum_{k=1}^{n} w_{2k} u_{2n-2k} = \zeta_{2n} + \sum_{k=1}^{n} c_{k-1} u_{2n-2k}.$$

Consequently,

$$\sum_{k=1}^{n} c_{k-1} u_{2n-2k} = u_{2n} - \zeta_{2n} = \frac{1}{2}\binom{2n}{n}. \tag{16}$$

Similarly one can show $\sum_{k=1}^{n} c_{k-1} u_{2n+1-2k} = u_{2n+1} - \zeta_{2n+1} = nc_n$.

Proof of Proposition 9.2. We show the proof via induction.

n = 1: ✓

Induction step: We need to distinguish between even and odd times. Assume first that the statement holds true for every $m < 2n$. We now show it for $2n$. To this end let

- q_{2k} = the number of paths below the x-axis until $2k-1$ and jumping to $+1$ at time $2k$.
- d_k = the number of paths starting in one and satisfying $x_k \geq 0$
- r_{2k+1} = the number of paths starting in one and hitting the x-axis for the first time at $2k+1$

Observe that

$$a_{2n} = \zeta_{2n} + \sum_{k=1}^{n} w_{2k} a_{2n-2k} + \sum_{k=1}^{n} q_{2k} d_{2n-2k} \tag{17}$$

for $n \geq 1$. One can show that $q_{2k} = c_{k-1}$ and $r_{2k+1} = c_k$. A combinatorial argument implies

$$d_{2k} = u_{2k} + \sum_{j=0}^{k-1} r_{2j+1} a_{2k-2j-1} = u_{2k} + \sum_{j=0}^{k-1} c_j a_{2k-2j-1}.$$

To sum up, Equation (17) can be simplified to

$$a_{2n} = \frac{1}{2}\binom{2n}{n} + \sum_{k=1}^{n} c_{k-1} a_{2n-2k} + \sum_{k=1}^{n} c_{k-1} u_{2n-2k} + \sum_{k=1}^{n} c_{k-1} \sum_{j=0}^{n-k-1} c_j a_{2(n-k)-2j-1}.$$

Note that Equation (16) implies $\sum_{k=1}^{n-1} c_{k-1} u_{2n-2k} = \frac{1}{2}\binom{2n}{n}$. With Equation (14) we obtain

$$\sum_{k=1}^{n} c_{k-1} \sum_{j=0}^{n-k-1} c_j a_{2(n-k)-2j-1} = \sum_{m=1}^{n-1} c_{n-m} a_{2m-1} = \sum_{k=1}^{n-1} c_k a_{2n-2k-1}.$$

Therefore

$$a_{2n} = \binom{2n}{n} + \sum_{k=1}^{n} c_{k-1} a_{2n-2k} + \sum_{k=1}^{n-1} c_k a_{2n-2k-1}.$$

With the induction hypothesis we obtain

$$\begin{aligned}
a_{2n} &= \binom{2n}{n} + \sum_{k=1}^{n} c_{k-1} \frac{1}{3}(2^{2n-2k+1}+1) + \sum_{k=1}^{n-1} c_k \frac{1}{3}(2^{2n-2k}-1) \\
&= \binom{2n}{n} + \sum_{k=0}^{n-1} c_k \frac{1}{3}(2^{2n-2k-1}+1) + \sum_{k=1}^{n-1} c_k \frac{1}{3}(2^{2n-2k}-1) \\
&= \binom{2n}{n} + c_0 \frac{1}{3}(2^{2n-1}+1) + \sum_{k=1}^{n-1} c_k \frac{1}{3}[(2^{2n-2k-1}+1) + 2^{2n-2k}-1] \\
&= \binom{2n}{n} + \frac{1}{3}(2^{2n-1}+1) + \sum_{k=1}^{n-1} c_k 2^{2n-2k-1}.
\end{aligned}$$

Equation (15) implies $\sum_{k=1}^{n-1} c_k 2^{2n-2k-1} = 2^{2n-1} - \binom{2n}{n}$, which further entails that $a_{2n} = \frac{1}{3}(2^{2n+1}+1)$.

We now make the induction step for odd times. Suppose that the statement is true for every $m \leq 2n$. Note that

$$a_{2n+1} \;=\; \zeta_{2n+1} + \sum_{k=1}^{n} w_{2k} a_{2n+1-2k} + \sum_{k=1}^{n} q_{2k} d_{2n+1-2k}. \tag{18}$$

One can show $d_{2k+1} = u_{2k+1} + \sum_{j=0}^{k} c_j a_{2k-2j}$. Therefore

$$a_{2n+1} \;=\; \binom{2n}{n} + \sum_{k=1}^{n} c_{k-1} a_{2n+1-2k} + \sum_{k=1}^{n} c_{k-1} u_{2n-2k+1} + \sum_{k=1}^{n} c_{k-1} \sum_{k=0}^{n-k} c_j a_{2(n-k)-2j}.$$

As above one can show that $\sum_{k=1}^{n} c_{k-1} \sum_{j=0}^{n-k} c_j a_{2(n-k)-2j} = \sum_{k=1}^{n} c_k a_{2n-2k}$. Hence, with the induction hypothesis,

$$\begin{aligned} a_{2n+1} &= \binom{2n}{n} + nc_n + \sum_{k=1}^{n} c_{k-1} a_{2n+1-2k} + \sum_{k=1}^{n} c_k a_{2n-2k} \\ &= \binom{2n}{n} + nc_n + \sum_{k=0}^{n-1} c_k a_{2n-2k-1} + \sum_{k=1}^{n} c_k a_{2n-2k} \\ &= \binom{2n}{n} + nc_n + \frac{1}{3}(2^{2n}-1) + c_n + \sum_{k=1}^{n-1} c_k 2^{2n-2k}. \end{aligned}$$

Since $\sum_{k=1}^{n-1} c_k 2^{2n-2k} = 2^{2n} - 2\binom{2n}{n}$, this shows

$$a_{2n+1} = \frac{1}{3}(2^{2n}-1) + 2^{2n} = \frac{1}{3}(2^{2n+2}-1).$$

□

10. Doubling the stake in each round

Zoe and Wael play yet another version of the game of heads and tails. Suppose that Zoe can choose, at the beginning of each round, how many Euros she wants to bet. Any bet in $\mathbb{Z}_{\geq 0}$ is allowed.

Assume that Zoe bets $\alpha(x)$ if the total gain of all the preceding rounds sums up to $x \in \mathbb{R}$. We refer to the function $\alpha : \mathbb{R} \to \mathbb{Z}_{\geq 0}$ as Zoe's betting rule. Note that if $\alpha(x) = 1 + 1_{(-\infty,0)}(x)$, then Zoe is playing the 1,2-strategy of Section 9.

Let Ω_n, $\mathcal{P}_n$, P_n and X_i, $i \in \{1,\ldots,n\}$, be defined as in Section 7. Let $G_0^\alpha = 0$ and define recursively

$$G_{n+1}^\alpha = G_n^\alpha + \alpha(G_n^\alpha) X_{n+1}.$$

G_n^α, $n \geq 0$, is the gain process of the betting rule α.

Question 9. Is there a betting rule α such that $\lim_n P_n(G_n^\alpha \geq 1) = 1$?

In the remainder we consider the following rule:

$$\alpha(x) = \begin{cases} 0, & \text{if } x > 0, \\ 1 + |x|, & \text{if } x \leq 0 \end{cases} \tag{19}$$

First observe that as soon as the total gain is > 0 Zoe stops betting. Therefore, under this stopping rule, once the gain process attains a positive value it remains there for the rest of the game.

In the first round Zoe bets $\alpha(0) = 1$ Euro. If heads turns up, then $G_1 = 1$ and hence also $G_n = 1$ for all $n > 1$.

Now suppose that the first toss is tails. Then $G_1 = -1$ and Zoe bets $\alpha(-1) = 2$ Euros in round 2. If the second toss is heads, then Zoe wins 2 Euros. In this case, $G_2 = -1 + 2 = 1$ and Zoe stops betting.

If the first two tosses turn up tails, then $G_2 = -1 - 2 = -3$ and Zoe bets $\alpha(-3) = 4$ in round 3. If she wins round three, then $G_3 = -3 + 4 = 1$, and she stops betting. If she loses round three, then $G_3 = -3 - 4 = -7$ and she bets 8 Euros in round 4, and so on.

With the first heads Zoe's gain process jumps to 1 and Zoe stopps betting thereafter. Therefore, Zoe's gain after round n is not equal to 1 if and only if the first n tosses all show tails. More precisely, $\{G_n \neq 1\} = \{(T, \dots, T)\}$, and hence $P_n(G_n = 1) = 1 - \frac{1}{2^n}$. In particular, $\lim_n P_n(G_n^\alpha \geq 1) = 1$.

Notice that the betting rule (19) implies that Zoe doubles the stake in each round as long as she has not won. Therefore α is also called a *doubling strategy*.

The gain process of a doubling strategy is not bounded from below. Hence an implementation requires an infinite amount of capital, and therefore doubling strategies do not work in practice. However, it happens every now and then that traders are tempted to follow doubling-like strategies in order to absorb trading losses. By doing so it can easily happen they accumulate large losses. At the moment where their banks can not provide them enough money for doubling, they are left with huge losses. Therefore, financial institutions try to make sure that their traders do not follow doubling strategies.

11. Zoe gets lucky against Wael

Once again, Zoe and Wael play the simple version of the coin tossing game: assume for instance that Zoe bets on tails and Wael bets on heads. Victory goes to the one that has the most winning bets after n coin tosses. Let $\Omega_n = \{H, T\}^n$, ω_i $(i \in \{1, \dots, n\})$, $\mathcal{P}_n$ and P_n be defined as in Section 9.

For every i we define a mapping $X_i : \Omega_n \to \{0, 1\}$ by setting

$$X_i(\omega) = \begin{cases} 0, & \text{if } \omega_i = H, \\ 1, & \text{if } \omega_i = T, \end{cases}$$

and

$$S_n(\omega) = X_1(\omega) + \cdots + X_n(\omega).$$

The random variable S_n counts the number of tails in the first n tosses, which is also the number of times that Zoe won the bet.

Assume that the probability of showing heads and the probability of showing tails are both equal to $\frac{1}{2}$. In that case, we expect intuitively that if n is large, say greater than 1000, the proportions of Zoe's and Wael's wins will be close to $\frac{1}{2}$. This result, known as the *law of large numbers*, says that, if the game is played during a sufficiently large period of time, the game will be *almost* tied, in the sense that winning proportions will be close to $\frac{1}{2}$, or said otherwise, that the winning margin will be small. However, we know from the arcsine law presented in Section 8, that either Zoe or Wael will be ahead most of the time.

Question 10. What is the probability that, after a large number coin tosses, Zoe gets lucky and wins 70% of the time? Said otherwise, what is the value of

$$P_n\left(\frac{S_n}{n} \geq 70\%\right) \text{ for large values of } n?$$

Let $a \in (\frac{1}{2}, 1]$ be fixed and let us compute

$$P_n\left(\frac{S_n}{n} \geq a\right) = P_n\left(S_n \geq a\,n\right) = \sum_{an \leq k \leq n} \binom{n}{k} \frac{1}{2^n},$$

which implies that

$$\frac{1}{2^n} \max_{an \leq k} \binom{n}{k} \leq P_n\left(\frac{S_n}{n} \geq a\right) \leq \frac{n+1}{2^n} \max_{an \leq k} \binom{n}{k}.$$

Then, using that $\binom{n}{\ell} \leq \binom{n}{m}$ for $0 \leq \ell \leq m \leq \lceil \frac{n}{2} \rceil$, we get

$$\frac{1}{2^n}\binom{n}{q} \leq P_n\left(\frac{S_n}{n} \geq a\right) \leq \frac{n+1}{2^n}\binom{n}{q}, \tag{20}$$

where we denoted $q := \lfloor an \rfloor + 1$. Taking the logarithm of the left hand side, dividing by n, and using Stirling's formula (Lemma B.1), we arrive at

$$\frac{1}{n}\log\frac{1}{2^n}\binom{n}{q} \sim -\log(2) + \frac{1}{n}\log\left(\frac{n}{n-q}\right)^{n-q} + \frac{1}{n}\log\left(\frac{n}{q}\right)^{q}. \tag{21}$$

Since $\frac{q}{n} = \frac{\lfloor an \rfloor + 1}{n}$ converges to a as n goes to infinity, the right hand side of (21) converges to

$$-\log(2) - (1-a)\log(1-a) - a\log(a).$$

The right hand side in (20) also converges to this quantity, by the same arguments. Thus we have proved the following :

Lemma 11.1. *For every $a \in (\frac{1}{2}, 1]$,*

$$\lim_{n\to+\infty} \frac{1}{n} \log P_n\left(\frac{S_n}{n} \geq a\right) = -I(a),$$

with $I(a) := a\log(a) + (1-a)\log(1-a) - \log(\frac{1}{2})$.

Notice that, by symmetry, the result above also holds for $a \in [0, \frac{1}{2})$. For $a \neq \frac{1}{2}$, the Lemma says that $P_n\left(\frac{S_n}{n} \geq a\right)$ converges to 0, and does so exponentially fast, roughly as $\exp(-nI(a))$. We can now give an approximate answer to Question 10: $P_n\left(\frac{S_n}{n} \geq 70\%\right)$ is approximately equal to

$$\begin{array}{rl} \exp(-100 * I(0.7)) \approx 0.00026, & \text{if } n = 100, \\ \exp(-1000 * I(0.7)) \approx 1.84 \times 10^{-36} & \text{if } n = 1000. \end{array}$$

A. Proof of the combinatorial results on paths

For the proof of Lemma 8.2 we use the following result.

Lemma A.1 (Reflection principle)**.** *Let $x \geq 0$. Then the number of paths in $\mathcal{Q}((0,0),(n,x))$ attaining the level -1 at some point is equal to $|\mathcal{Q}((0,-2),(n,x))|$.*

Proof. Let f be a path in $\mathcal{Q}((0,0),(n,x))$ that attains the level -1, and let $\sigma = \min\{k : f(k) = -1\}$ be the first time where f takes the value -1. Now reflect the path up to σ along the horizontal level -1. More precisely, assign to f the path $\widehat{f} \in \mathcal{Q}((0,-2),(n,x))$ satisfying

$$\widehat{f}(j) = \begin{cases} -(f(j)+1)) - 1, & \text{for } j < \sigma, \\ f(j), & \text{for } j \geq \sigma. \end{cases} \tag{22}$$

Note that if $f, g \in \mathcal{Q}((0,0),(n,x))$ attain -1 and $f \neq g$, then $\widehat{f} \neq \widehat{g}$. Moreover, for any $h \in \mathcal{Q}((0,-2),(n,x))$ there exists an $f \in \mathcal{Q}((0,0),(n,x))$ such that $\widehat{f} = h$. In other words, the mapping $f \mapsto \widehat{f}$ is a one-to-one correspondance between the set of paths in $\mathcal{Q}((0,0),(n,x))$ reaching -1 and the set $\mathcal{Q}((0,-2),(n,x))$. Therefore, both sets have the same number of elements. $\square$

Proof of Lemma 8.2. Let B be the set of non-negative paths in $\mathcal{Q}_{2n}(0,0)$, and B_k the set of non-negative paths in $\mathcal{Q}((0,0),(2n,2k))$. Notice that if $f \in \mathcal{Q}_{2n}(0,0)$, then $f(2n)$ is an even number. Therefore, $B = \cup_{k=0}^{n} B_k$.

The set B_n contains only one path. In order to calculate $|B_k|$ for $k \in \{0,\ldots,n-1\}$, observe first that the set B_k is $\mathcal{Q}((0,0),(2n,2k))$ minus the paths reaching -1 at some point. Therefore, the reflection principle and Lemma 8.1 yield

$$|B_k| = |\mathcal{Q}((0,0),(2n,2k))| - |\mathcal{Q}((0,-2),(2n,2k))| = \binom{2n}{n-k} - \binom{2n}{n-(k+1)}. \tag{23}$$

Summing up all $|B_k|$ we obtain a telescoping sum and hence

$$|B| = \sum_{k=0}^{n} |B_k| = 1 + \sum_{k=0}^{n-1} \left(\binom{2n}{n-k} - \binom{2n}{n-(k+1)} \right) = \binom{2n}{n}.$$

$\square$

B. Proof of Stirling's formula

Lemma B.1. *For all $n \in \mathbb{N}$ we have*

$$e^{\frac{11}{12}} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e n^{n+\frac{1}{2}} e^{-n}. \tag{24}$$

Moreover,

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}. \tag{25}$$

Proof. Let $d_n = \log(n!) - (n + \frac{1}{2}) \log(n) + n$. Then we have

$$n! = e^{d_n} n^{n+\frac{1}{2}} e^{-n}. \tag{26}$$

Observe that

$$d_n - d_{n+1} = (n + \frac{1}{2}) \log(1 + \frac{1}{n}) - 1.$$

Recall the expansion $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, for $x > 0$. This entails $\log(\frac{1}{1-x}) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$, and hence $\log(\frac{1+x}{1-x}) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$. Note that $1 + \frac{1}{n} = \frac{1+1/(2n+1)}{1-1/(2n+1)}$ and hence

$$d_n - d_{n+1} = \frac{1}{3} \frac{1}{(2n+1)^2} + \frac{1}{5} \frac{1}{(2n+1)^4} + \cdots.$$

This further yields

$$d_n - d_{n+1} \leq \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{(2n+1)^{2k}} = \frac{1}{3} \frac{1}{(2n-1)^2 - 1} = \frac{1}{12n} - \frac{1}{12(n+1)}. \tag{27}$$

Note that (27) shows that $(d_n)_{n \geq 1}$ is decreasing and $(d_n - \frac{1}{12n})_{n \geq 1}$ is increasing. In particular, $d_n \leq d_1 = 1$ and $d_n \geq d_1 - \frac{1}{12} = \frac{11}{12}$. Together with (26) this implies

$$e^{\frac{11}{12}} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e n^{n+\frac{1}{2}} e^{-n}.$$

Finally, one can show that d_n converges to $\log(\sqrt{2\pi})$ (see e.g. [5] for details) and hence we obtain (13). $\square$

C. Implementation of recursion formula (4) in Python

```
import numpy as np

def A(n,x):
    a=np.zeros(n+1)
    for m in range(n+1):
        if m <= x:
            a[m]=2**m
        else:
            for j in range(1,x+2):
                a[m] = a[m]+a[m-j]
    return a[n]

def P(n,x):
    return A(n,x)/(2**n)
```

Notes

The idea of the proof presented in Section 1 and Section 3 can be found in [6].

The account of the arcsine law in Section 8 is inspired by the book of Feller [2] and Lesigne [5].

The question of Section 9 is owed to our colleague Ingo Althöfer. See [1] for an article on the 1,2-random walk.

The recursion formula (4) is drawn from the article [7], which provides a more extensive analysis of long head runs.

Section 5 is inspired by Chapter 5 in [3].

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