Prony-type methods for recovery of and approximation by exponential sums

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We introduce a new method - ESPIRA (Estimation of Signal Parameters via Iterative Rational Approximation) [1] - for the recovery of complex exponential sums

$$f(t) = \sum_{j=1}^{M} \gamma_j \mathrm{e}^{\lambda_j t},$$

that are determined by a finite number of parameters: the order M, weights $\gamma_j \in \mathbb{C} \setminus \{0\}$ and nodes $\mathrm{e}^{\lambda_j} \in \mathbb{C}$ for $j = 1, ..., M$. Our new recovery procedure is based on the observation that Fourier coefficients (or DFT coefficients) of exponential sums have a special rational structure. To reconstruct this structure in a stable way we use the AAA algorithm proposed by Nakatsukasa et al. [2]. We show that ESPIRA can be interpreted as a matrix pencil method applied to Loewner matrices. During the talk we will demonstrate that ESPIRA outperforms Prony-type methods such as ESPRIT for noisy data and for signal approximation by short exponential sums. We are specially interested in exponential approximation of the Gaussians.

Furthermore we propose a method to approximate the Gaussian function on $\mathbb{R}$ by a short exponential sum in the weighted space $L_2(\mathbb{R}, \mathrm{e}^{-t^2/2\rho})$ [3]. We prove that the optimal frequency parameters $\lambda_1, \dots, \lambda_M$ for this method in the approximation problem $\min\limits_{\lambda_1,\dots,\lambda_M,\gamma_1\dots\gamma_M} \|\mathrm{e}^{-\cdot^2/2\sigma} - \sum\limits_{j=1}^{M} \gamma_j \, \mathrm{e}^{\lambda_j \cdot}\|_{L_2(\mathbb{R},\mathrm{e}^{-t^2/2\rho})}$, are zeros of a scaled Hermite polynomial. This observation leads us to a numerically stable approximation method with low computational cost of only $\mathcal{O}(M^3)$ operations. Furthermore, we derive a direct algorithm to solve this approximation problem based on a matrix pencil method for special structured matrices. The entries of these matrices are determined by hypergeometric functions.

The talk will be based on joint work with Gerlind Plonka and Markus Petz (University of Göttingen).

References

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[3] N. Derevianko, G. Plonka, Differential approximation of the Gaussian by short cosine sums with exponential error decay, https://arxiv.org/abs/2307.13587.