Diffusion control ranking games
Part I
Diffusion control with threshold rewards

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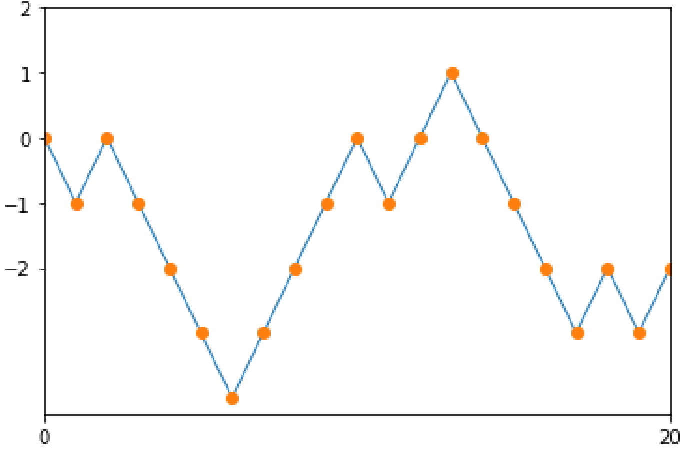
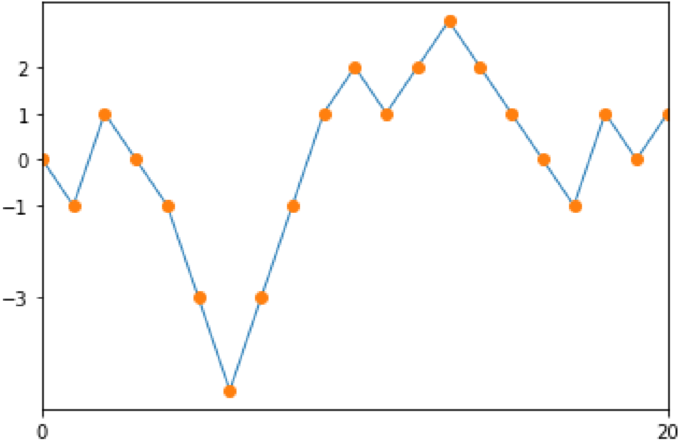
The simple random walk



The 1-2 random walk



Question: $\lim_{N\to\infty} P(X_N \geq 0) = ?$

The 1-2 random walk

Let (Y_n) be an iid sequence with $P(Y_n = \pm 1) = \frac{1}{2}$.
The 1-2 random walk (X_n) is defined by $X_0 = 0$ and

$$X_{n+1} = \begin{cases} X_n + Y_n, & \text{if } X_n \geq 0, \\ X_n + 2Y_n, & \text{if } X_n < 0. \end{cases}$$

Lemma

The number of 1-2 paths of length n with a value ≥ 0 at the end is given by

$$\frac{1}{3}(2^{n+1} + (-1)^n).$$

For a proof see Chapter 9 in the unpublished notes "Lessons from coin tosses" [AKT].

Corollary

$\lim_n P(X_n \geq 0) = \frac{2}{3}$.

Why interesting ?

- different step sizes in different regions changes the skewness and the quantiles (but not the mean)
- taking risk in the loosing region increases the probability of getting back into the winning region
- explains behavior of sports teams, managers, animals searching for food, ...

Questions

- Suppose a controller can choose the step size from the set $\{1, 2\}$ at any time. How can one prove that the feedback control (closed loop control)

$$\sigma^*(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 2, & \text{if } x < 0, \end{cases}$$

  maximizes the probability of being above 0 at some time finite time N ?
- What about the x-y random walk, where $x, y \in (0, \infty)$?

Concise answers can be found for the

diffusion limits of the random walks

Diffusion limit of the simple random walk

Scaling: time with N, space with $1/\sqrt{N}$

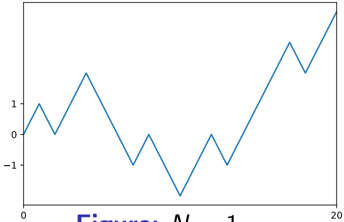


Figure: $N = 1$

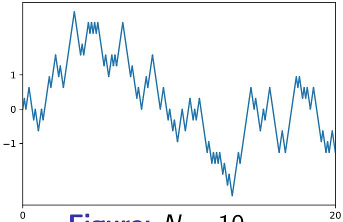


Figure: $N = 10$

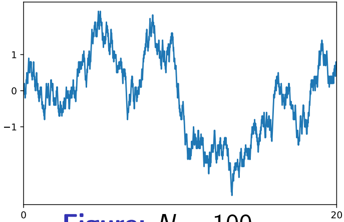


Figure: $N = 100$

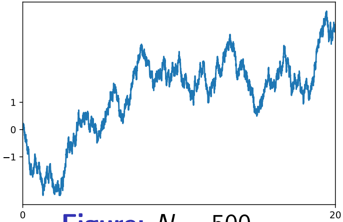


Figure: $N = 500$

Diffusion limit the simple random walk

Let $(Y_n)_{n\in\mathbb{N}_0}$ be the simple random walk. Scaled version:

$$Y_t^{(N)} := \begin{cases} \frac{1}{\sqrt{N}} Y_{Nt}, & \text{if } Nt \in \mathbb{Z}_{\geq 0}, \\ \text{linear} & \text{else} . \end{cases}$$

Theorem (Donsker's Theorem)

Let $T \in (0, \infty)$. Then $(Y_t^{(N)})_{t\in[0,T]}$ converges in distribution to a Brownian motion $(W_t)_{t\in[0,T]}$.

Diffusion limit of the 1-5 random walk

Scaling: time with N, space with $1/\sqrt{N}$

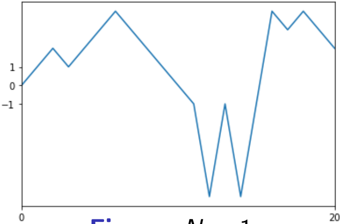


Figure: $N = 1$

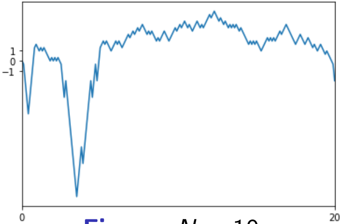


Figure: $N = 10$

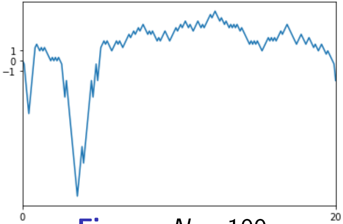


Figure: $N = 100$

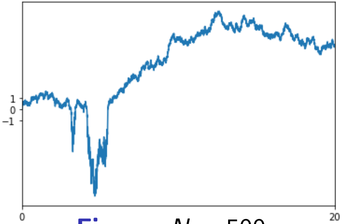


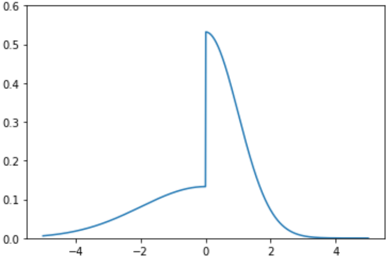
Figure: $N = 500$

Oscillating Brownian motion

An OBM is a process solving an SDE of the form

$$dX_t = \left(\sigma_1 1_{\{X_t \geq b\}} + \sigma_2 1_{\{X_t < b\}}\right) dW_t$$

- formula for the density in closed form



- Link to skew BM

References

- Keilson, Wellner 1978
- McNamara 1983

CLT for the σ_1-σ_2 random walk

Let $(X_n)_{n\in\mathbb{N}_0}$ be the σ_1-σ_2 random walk. Scaled version:

$$X_t^{(N)} := \begin{cases} \frac{1}{\sqrt{N}} X_{Nt}, & \text{if } Nt \in \mathbb{Z}_{\geq 0}, \\ \text{linear} & \text{else .} \end{cases}$$

Theorem

Let $T \in (0,\infty)$. Then $(X_t^{(N)})_{t\in[0,T]}$ converges in distribution to the OBM with parameters σ_1 and σ_2.

Diffusion control problems

A diffusion control problem

- state dynamics:

$$dX_t^\alpha = \alpha_t dW_t$$

- controls α with values in $[\sigma_1, \sigma_2]$, where $0 < \sigma_1 < \sigma_2$
- target:

$$P(X_T^\alpha > 0) \longrightarrow max!$$

Questions:

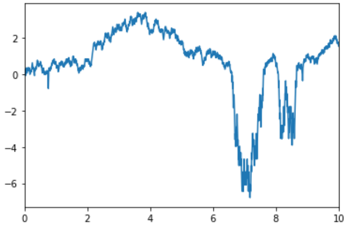
1. Optimal control = ?
2. maximal probability if $X_0 = 0$?

Solution of the control problem

Theorem
The control with feedback function

$$\sigma^*(x) = \begin{cases} \sigma_1, & \text{if } x \geq 0, \\ \sigma_2, & \text{if } x < 0, \end{cases}$$

is optimal.



Moreover,

$$\max_{\alpha} P(X_T^{\alpha} > 0 | X_0^{\alpha} = 0) = \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

Analytic solution method

State dynamics:

$$dX_t^\alpha = \alpha_t dW_t$$

Gain function:

$$J(t, x, \alpha) = P(X_T^{t,x,\alpha} \geq 0),$$

Value function:

$$V(t, x) = \sup_{\alpha \in \mathcal{A}} J(t, x, \alpha).$$

HJB equation:

$$-w_t(t, x) - \frac{1}{2} \sup_{u \in [1,2]} u^2 w_{xx}(t, x) = 0,$$

Terminal condition: $w(T, x) = 1_{[0,\infty)}(x).$

Analytic solution method II

A **candidate solution** of the HJB equation:

$$F(t,x) = \begin{cases} \frac{2}{3} + \frac{1}{3}\int_{T-t}^{\infty} \frac{|x|}{\sqrt{2\pi z^3}} e^{-\frac{|x|^2}{2z}}\,dz, & x > 0, \\ \frac{2}{3}, & x = 0, \\ \frac{2}{3}\int_0^{T-t} \frac{|x/2|}{\sqrt{2\pi z^3}} e^{-\frac{|x/2|^2}{2z}}\,dz, & x < 0. \end{cases}$$

Classical verification yields

Theorem

$V(t,x) = F(t,x)$, and the optimal control function is given by $\alpha(t,x) = \sigma^*(x)$.

The inverse control problem of McNamara 1983

Suppose that controlled state dynamics satisfy

$$dX_t^\alpha = \alpha_t dW_t$$

with $\alpha \in [\sigma_1, \sigma_2]$, and let

$$\sigma^*(x) = \begin{cases} \sigma_1, & \text{if } x \geq 0, \\ \sigma_2, & \text{if } x < 0, \end{cases}$$

be threshold control function from before.
Terminal reward: $E[R(X_T^\alpha)]$.

McNamara's pb: For which reward functions R the threshold strategy σ^* is optimal?

The inverse control problem cont'd

We know already one example: $R(x) = 1_{[0,\infty)}(x)$.

Theorem (McNamara 1983)

Let R be a continuous function with $R(0) = 0$. Then σ^* is an optimal feedback control if and only if

$$
\begin{aligned}
(i)&\ R \text{ is convex on } (-\infty, 0) \text{ and concave on } (0, \infty),\\
(ii)&\ \sigma_2 R(\sigma_1 x) = -\sigma_1 R(-\sigma_2 x) \quad x \geq 0.
\end{aligned}
$$

Example:

$$R(x) = \sqrt{\sigma_1 x}1_{(0,\infty)}(x) - \sqrt{\sigma_2 |x|}1_{(-\infty,0)}(x)$$

Math lessons to take

- Discrete time processes are usually more difficult to analyze than continuous time counterparts. Therefore: to simplify discrete time models, you can pass to the limit.
- Diffusion control problems in cont. time easier than corresponding control problems in discrete time.
- Standard approach for solving control problems: set up the HJB equation, try to find an explicit solution and then do a verification.
- Diffusion control allows to change skewness and quantiles (but not the mean)

Economic lessons to take

- Threshold rewards incentivize agents to take risk if things are going badly
- We confirm a known rule from sports: take risk if behind, play safe if ahead

What if the payoff depends on the performance of other agents?

- sports games: a team wins if it has more points than the other team
- management: bonus if the own company performs better than other companies
- research: the best results will be published or put into production
- elections: a candidate is elected if she has more votes than another candidate

——> see Part II

Literature

- S. Ankirchner and N. Kazi-Tani. *Lessons from coin tosses*. Unpublished lecture notes. Available at: `https://www.fmi.uni-jena.de/2376/prof-dr-stefan-ankirchner`
- McNamara, J. M. *Optimal control of the diffusion coefficient of a simple diffusion process.* Mathematics of Operations Research 8.3 (1983): 373-380.