Diffusion control ranking games
Part II

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A 2-player game

- State of player 1:

$$dX_t = \alpha(X_t, Y_t)dW_t^1, \quad X_0 = 0$$

- State of player 2:

$$dY_t = \beta(X_t, Y_t)dW_t^2, \quad Y_0 = 0$$

- $\alpha, \beta : \mathbb{R}^2 \to [\sigma_1, \sigma_2]$ closed loop controls
- W^1 and W^2 are independent BM

2-player game cont'd

$$\text{reward of player 1} = \begin{cases} 1, & \text{if } X_T > Y_T, \\ 0, & \text{else.} \end{cases}$$

$$\text{reward of player 2} = \begin{cases} 1, & \text{if } Y_T > X_T, \\ 0, & \text{else.} \end{cases}$$

Comments:

- Zero-sum payoff
- Players can observe the opponent's state

Solving the game

The state difference $D_t := X_t - Y_t$ satisfies

$$dD_t = (\alpha_t^2 + \beta_t^2)^{1/2} d\tilde{W}_t$$

Target of player 1: $P(D_T > 0) \longrightarrow \max!$
Target of player 2: $P(D_T > 0) \longrightarrow \min!$

Irrespective of β_t:

- $\alpha_t = \sigma_2$ maximizes the diffusion rate
- $\alpha_t = \sigma_1$ minimizes the diffusion rate

Solving the game

Theorem

Let

$$\alpha^*(x,y) = \begin{cases} \sigma_1, & \text{if } x \geq y, \\ \sigma_2, & \text{if } x < y, \end{cases}$$

and

$$\beta^*(x,y) = \alpha^*(y,x).$$

Then (α^*, β^*) *is a saddle point (and hence a Nash equilibrium).*

Rigoros proof

Isaacs equations....

What if more than 2 players?

- management: bonus if the own company is among the best performing companies
- research competition among many R & D teams
- Olympic games: the best three athletes receive a medal
- elections with many candidates

A large ranking game

- n players
- $X^1, \ldots, X^n$ = states of player $1, \ldots, n$

$$dX_t^{i,a} = a_i(X_t^{i,a}, X_t^{-i,a})dW_t^i, \quad X_0^{i,a} = 0.$$

- $a_i : \mathbb{R}^n \to [\sigma_1, \sigma_2]$
- $W^1, \ldots, W^n$ independent Brownian motions

A large ranking game

- empirical distribution of all states at T:

$$\mu^{n,a} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_T^{i,a}}$$

- the $\alpha * 100\%$ best players receive a prize:

$$\text{reward of player } i = \begin{cases} 1, & \text{if } X_T^{i,a} > (1-\alpha)\text{-quantile of } \mu^{n,a}, \\ 0, & \text{else.} \end{cases}$$

How to find / characterize Nash equilibria?

Stochastic games with many players in general are difficult to analyze. Additional difficulty here: discontinuous reward function.

Idea: for large n the mean-field game version yields an approximate Nash equilibrium.

The mean field approximation

- 1 player
- state dynamics

$$dX_t = a(X_t)dW_t, \quad X_0 = 0$$

- $a : \mathbb{R} \to [\sigma_1, \sigma_2]$
- a^* is an equilibrium strategy if

$$P(X_T^{a^*} > q(X_T^{a^*}, 1-\alpha)) = \max_a P(X_T^a > q(X_T^{a^*}, 1-\alpha))$$

The mean field equilibrium is a threshold strategy

Theorem

The threshold strategy with threshold

$$b^* = \begin{cases} -\sigma_2\sqrt{T}\Phi^{-1}\left(\frac{\alpha(\sigma_1+\sigma_2)}{2\sigma_2}\right), & \text{if } \alpha \le \frac{\sigma_2}{\sigma_1+\sigma_2}, \\ \sigma_1\sqrt{T}\Phi^{-1}\left(\frac{(1-\alpha)(\sigma_1+\sigma_2)}{2\sigma_1}\right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1+\sigma_2}, \end{cases} \tag{1}$$

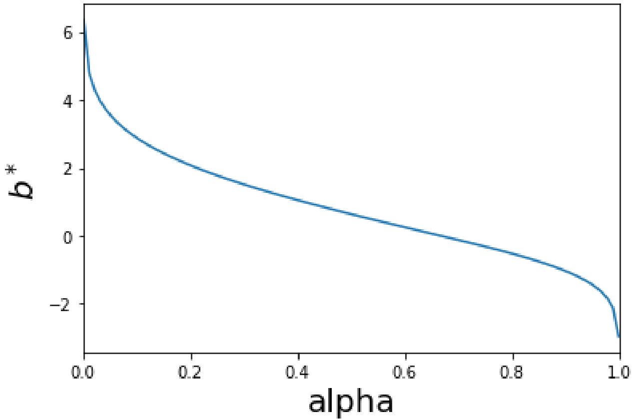
is an equilibrium strategy for the mean field game.

The mean field equilibrium is an approximate equilibrium of the n player game

Theorem

Let $a = (a_1, \ldots, a_n)$ be the tuple of mean-field equilibrium strategies. Then there exists $\varepsilon_n \downarrow 0$ such that $a = (a_1, \ldots, a_n)$ is an ε_n-Nash equilibrium of the n-player game. We can choose $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$.

The smaller the cake...



The smaller the cake...

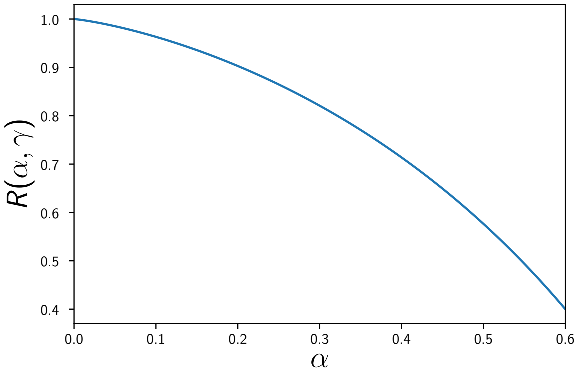
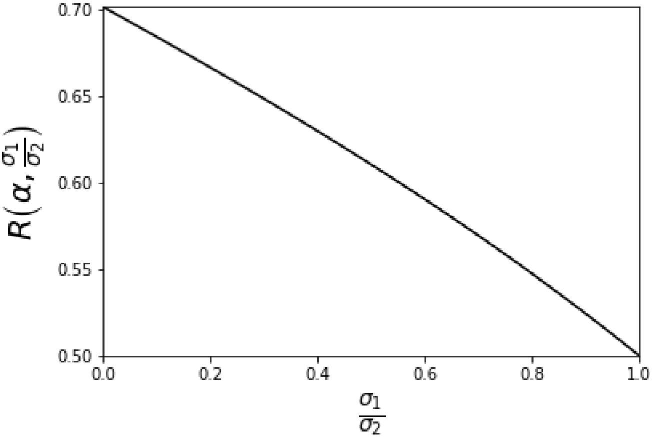


Figure: $\gamma := \frac{\sigma_1}{\sigma_2} = \frac{2}{3}$, e.g. $\sigma_1 = 2, \sigma_2 = 3$

$$R(\alpha) = \frac{1}{T} \int_0^T P(X_t^{b^*} < b^*)\, dt$$

The larger the scope of action...



The heat of the battle

Part of the players choosing the small volatility σ_1

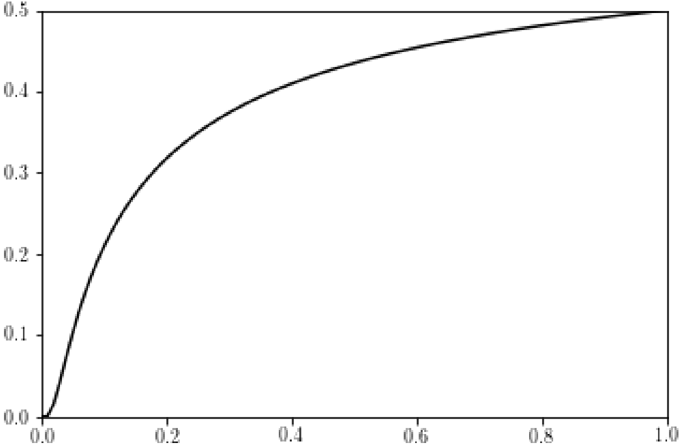


Figure: Parameters: $\sigma_1 = 1$, $\sigma_2 = 2$, $T = 1$, $\alpha = 0.5$

Comparison of large games with the 2-player game

2 players	∞ players
only relative position counts	only absolute position counts
observability is crucial	observability is irrelevant
one of the players always chooses σ_1	at the beginning all choose σ_2

Conclusion

- Closed form equilibrium for the limiting cases $n = 2$ and $n = \infty$.
- Games with $n \geq 3$ players: equilibrium strategies depend on both the relative and absolute position

Literature

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