Difference approximation for equations with interaction

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This study investigates stochastic differential equations (SDEs) with interaction, introduced by A.A. Dorogovtsev to model the evolution of large systems of interacting particles in random environments. These equations capture the mutual dependence of particle trajectories on the overall mass distribution. The system is described by the equation:

$$dx(u,t) = a(x(u,t), \mu_t)\,dt + \int_{\mathbb{R}^d} b(x(u,t), \mu_t, p)\,W(dp, dt),$$

where $x(u,0) = u$ for $u \in \mathbb{R}^d$, and $\mu_t = \mu_0 \circ x(\cdot, t)^{-1}$. Here, μ_0 is the initial mass distribution, W is a Wiener sheet, and the coefficients a and b are Lipschitz continuous with respect to their arguments. These properties ensure the well-posedness of the solutions under the Wasserstein distance.

The primary goal is to construct and analyze difference approximation schemes for these equations, addressing challenges related to their infinite-dimensional and nonlinear nature.

For two probability measures μ_1, μ_2, the Wasserstein distance of order n is defined as:

$$\gamma_n(\mu_1, \mu_2) = \inf_{\zeta \in \Gamma(\mu_1, \mu_2)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |u - v|^n \,\zeta(du, dv) \right)^{1/n},$$

where $\Gamma(\mu_1, \mu_2)$ is the set of all couplings of μ_1 and μ_2. Using the Kolmogorov-Totoki theorem the approximation error is bounded by:

$$E \sup_{u \in K} \sup_{t \in [0,1]} |x(u,t) - x_N(u,t)| \le C \gamma_2(\mu_0, \mu_N),$$

where γ_2 denotes the second-order Wasserstein distance.

Spatial discretization is achieved by dividing the domain K into N^d grid cells and defining an empirical measure as:

$$\mu_N = \sum_{i_1, \ldots, i_d} \alpha_{i_1, \ldots, i_d} \delta_{u_{i_1, \ldots, i_d}},$$

where the weights $\alpha_{i_1, \ldots, i_d}$ correspond to the mass in each grid cell. The Wasserstein distance between the initial measure μ_0 and its discretized version μ_N is estimated as:

$$\gamma_2(\mu_0, \mu_N) \le \frac{d}{N^2},$$

ensuring that the spatial discretization error decreases with finer grids.

References

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