Analysis on graphs

Course given at TU Graz March 2013 **Prof. Dr. D. Lenz**

Exercise Sheet III **Due to a later point of time**

(1) Let (b, c) be a graph over X and Q the associated Dirichlet form. Let $K \subset X$ be non-empty. Show that the restriction of Q to the functions on K is again a Dirichlet form and compute the corresponding b_K and c_K.

(2) Let (b, c) be a graph over X and Q the associated Dirichlet form. Let $m : X \longrightarrow (0, \infty)$ be given and define the inner product $\langle \cdot, \cdot \rangle_m$ on the space of real-valued functions on X by

$$\langle f, g \rangle_m := \sum_{x \in X} f(x) g(x) m(x).$$

Denote the arising Hilbert space by $\ell^2(X, m)$.

(a) Determine the operator Δ_m on $\ell^2(X, m)$ satisfying

$$\langle \Delta_m f, g \rangle_m = Q(f, g)$$

for all f and g.

(b) Find the smallest eigenvalue of Δ_m in the case that $c = 0$.

(3) Let (b, c) be a graph over X with $c = 0$ and define $m : X \longrightarrow (0, \infty)$ by $m(x) := \sum_y b(x, y)$ and consider the operator Δ_m on $\ell^2(X, m)$ introduced in the previous exercise. Define for $K \subset X$

$$S_K := \sum_{x \in K, y \notin K} b(x, y), \;\; A_K = \sum_{x \in K} m(x).$$

(a) Show that $S_K = Q(1_K) = -Q(1_K, 1_{X \setminus K})$ and $A_K = \|1_K\|_m^2$, where 1_K denotes the characteristic function of K and conclude that

$$\alpha := \min\{\frac{S_K}{A_K} : 0 < K \leq \frac{A_X}{2}\} = \min\{\frac{Q(1_K)}{\|1_K\|_m^2} : 0 < K \leq \frac{A_X}{2}\}.$$

(c) Show that the second smallest eigenvalue λ_2 of Δ_m satisfies $\lambda_2 \geq \frac{\alpha^2}{2}$.